1 Homework Solutions

18.335 - Fall 2004

1.1 Let A be an orthogonal matrix. Prove that $|det(A)| = 1$. Show that if B is also orthogonal and $det(A) = -det(B)$, then $A + B$ is singular.

$(\det A)^2 = \det A \det A = \det A \det A^T = \det AA^T = \det I = 1$

$A + B$ is singular iff $A^T(A + B) = I + A^T B$ is. $A^T B$ is orthogonal so all its eigenvalues are 1 or -1. Since their product is equal to $\det A^T B = -1$ then at least one of the eigenvalues of $A^T B$ must be -1. Let the corresponding vector be x. Then $(I + A^T B)x = x - x = 0$, so $I + A^T B$ is singular and so is $A + B$.

Second proof: $\det(A + B) = -\det(A^T)\det(A + B)\det(B^T) = -\det(A^T AB^T + A^T BB^T) = -\det(A^T + B^T) = -\det(A + B)$, so $\det(A + B) = 0$.

1.2 Trefethen 2.5

(a) Let λ be an eigenvalue of S and v its corresponding eigenvector so that $Sv = \lambda v \Rightarrow v^* Sv = \lambda v^* v = \lambda \|v\|^2$. We also have $\overline{v^* Sv} = v^* S^* v = -v^* Sv$. This implies that $\bar{\lambda} = -\lambda \Rightarrow \lambda$ is imaginary.

(b) If $(I - S)\, v = 0$ for $v \neq 0$ then $Sv = v$ and this means that 1 is an eigenvalue of S, a contradiction to (a).

(c) We have:

$$\begin{aligned}
Q^* Q &= \left[(I - S)^{-1}(I + S)\right]^* (I - S)^{-1}(I + S) \\
&= (I + S^*)(I - S^*)^{-1}(I - S)^{-1}(I + S) \\
&= (I - S)(I + S)^{-1}(I - S)^{-1}(I + S) \\
&= (I + S)^{-1}(I - S)(I - S)^{-1}(I + S) = I
\end{aligned}$$

where we have used that if $AB = BA$ and B is invertible that $AB^{-1} = B^{-1}A$

1.3 Trefethen 3.2

We know that $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$. Choose an eigenvalue λ of A and let $x_\lambda \neq 0$ such that $Ax_\lambda = \lambda x_\lambda$. Then $\frac{\|Ax_\lambda\|}{\|x_\lambda\|} = \frac{\|\lambda x_\lambda\|}{\|x_\lambda\|} = \frac{|\lambda|\,\|x_\lambda\|}{\|x_\lambda\|} = \lambda$. Thus we have $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \geq |\lambda|$. So $\|A\| \geq |\lambda|$ and since this is true for any eigenvalue of A we get $\|A\| \geq \sup\{|\lambda|, \lambda \text{ eigenvalue of } A\} = \rho(A)$.

1.4 Trefethen 3.3

(a) By definition $\|x\|_\infty = \max_{1\le i\le m} |x_i| \le \sqrt{\sum_{j=1}^{m} \left|x_j^2\right|} = \|x\|_2$. Equality is achieved when we have a vector with only one non-zero component.

(b) Again, using the definition $\|x\|_2 = \sqrt{\sum_{j=1}^{m} \left|x_j^2\right|} \le \sqrt{m \max_{1\le i\le m} |x_i|} = \sqrt{m}\,\|x\|_\infty$.

We have equality for a vector whose components are equal to each other.

(c) Denoting by r_j the j-th row of A we have $\|A\|_\infty = \max_{1\le j\le m} \|r_j\|_1$. For some vector $v \in \mathbb{C}^n$, $v^* = (1,...,1)/\sqrt{n}$ and using the 2-norm definition we get $\|A\|_2 = \sup_{|x|=1} \|Ax\|_2 \ge \|Av\|_2 = \frac{1}{\sqrt{n}}\sqrt{\sum_{j=1}^{m} \|r_j\|_1^2}$. These yield $\|A\|_\infty = \max_{1\le j\le m} \|r_j\|_1 \le \sqrt{\sum_{j=1}^{m} \|r_j\|_1^2} \le \sqrt{n}\,\|A\|_2$. Equality is achieved for a matrix which is zero everywhere except along a row of ones.

(d) Using the notation from part (c), $\|A\|_2 = \frac{1}{\sqrt{n}}\sqrt{\sum_{j=1}^{m} \|r_j\|_1^2} \le \sqrt{\sum_{j=1}^{m} \|r_j\|_1^2} \le \sqrt{m} \max_{1\le j\le m} \|r_j\|_1 = \sqrt{m}\,\|A\|_\infty$. We get equality for a square matrix which is zero everywhere except along a column of ones.

1.5 Prove that $\|xy^*\|_F = \|xy^*\|_2 = \|x\|_2\,\|y\|_2$ for any x and $y \in \mathbb{C}^n$.

$\|xy^*\|_F = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} |x_i \bar{y}_j|^2} = \sqrt{\sum_{i=1}^{n} |x_i|^2}\sqrt{\sum_{j=1}^{n} |\bar{y}_j|^2} = \|x\|_2\,\|y\|_2$

$\|xy^*\|_2 = \sup_{z\in\,\mathbb{C}^n} \frac{\|xy^*z\|_2}{\|z\|_2} = \sup_{z\in\,\mathbb{C}^n} \frac{\|x\|_2\,|y^*z|}{\|z\|_2}$. This ratio is maximized if $z//y$, so that $|y^*z| = \|y\|_2^2$, thus completing the proof.