18.330 :: Homework 1 :: Spring 2012 :: Due February 23

1. (3 pts) Consider the convergent series[1]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

   (a) Is this series absolutely convergent or conditionally convergent? Justify your answer in a rigorous fashion.

   (b) Write a short program (e.g. in Matlab) to compute partial sums. How many terms are needed to obtain 3 correct digits of $\pi/4$?

2. (3 pts) Consider the geometric series $1 + x + x^2 + x^3 + \dots$ and its partial sums

$$S_N = \sum_{n=0}^{N} x^n.$$

   (a) When $x \neq 1$, show by induction that

$$S_N = \frac{1 - x^{N+1}}{1 - x}.$$

   (b) While the formula above is valid for all $x \neq 1$, it only implies convergence of the geometric series to the limit $\frac{1}{1-x}$ in the case $|x| < 1$. Explain why the geometric series is always divergent when $|x| > 1$.

   (c) Find a formula for S_N when $x = 1$.

3. (4 pts) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^q}.$$

   (a) Show rigorously that the series converges when $q > 1$, by comparing it to an integral.

   (b) Show rigorously that the series diverges when $0 < q < 1$, by comparing it to an integral.

4. (Bonus, 1 pt. Bonus questions do not bring your score above 10/10.) It turns out that the following series converges for all real x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

   where $n! = n \cdot (n-1) \cdot (n-2) \dots 2 \cdot 1$ is the factorial of n. Write a program to compute the partial sum S_N for $x = -25$, and various values of N. How do you explain that your answer is never close to e^{-25}, regardless of how you choose N?

[1] It follows from the alternating series test that the series is convergent. One way to check that the limit is $\pi/4$ is to consider the Taylor series of `atan` centered at $x = 0$, and evaluate it at $x = 1$.