18.330 Handout Mon 01 Mar 04

Some Numerical Fun with Euler/Maclaurin

1 $$\boxed{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx}$$ via

$$\sqrt{2\pi}\; S(h) = h \sum_{n=-\infty}^{\infty} e^{-n^2h^2/2}$$

h	$S(h)$	$h^2 \cdot \ln(\text{error})$
2.00	1.014383772062230	-16.97
1.90	1.008439960755230	-17.24
1.80	1.004520190212421	-17.49
1.70	1.002161335688068	-17.74
1.60	1.000896079646551	-17.96
1.50	1.000309722287338	-18.18
1.40	1.000084574503106	-18.38
1.30	1.000016922642704	-18.57
1.20	1.000002227486564	-18.74
1.10	1.000000164517561	-18.90
1.00	1.000000005350576	-19.05
.90	1.000000000052185	-19.18
.80	1.000000000000081	-19.30

2 $$\boxed{\int_0^{\pi/2} (1 + 3\sin^2\theta)^{1/2}\,d\theta}$$

via (n+1) trapezoidal samplings at $\theta = 0, h, 2h, \ldots, nh = \pi/2$

n	T_n	error
1	2.356194490192	-.065917564945
2	2.419920778321	-.002191276816
3	2.421985353498	-.000126701639
4	2.422103097548	-.000008957588
5	2.422111351478	-.000000703659
6	2.422111996131	-.000000059006
7	2.422112049963	-.000000005173
8	2.422112054668	-.000000000469
9	2.422112055093	-.000000000043
10	2.422112055133	-.000000000004
11	2.422112055137	.000000000000
12	2.422112055137	.000000000000

3 $$\boxed{\frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{3}{5 - 4\cos\theta}\,d\theta}$$

again via trapezoidal rule T_n, followed by 3 rounds of Aitken/Shanks extrapolation

n	T_n	S^1	S^2	S^3
1	3.000000000000			
2	1.666666666667			
3	1.285714285714	1.133333333333		
4	1.133333333333	1.031746031746		
5	1.064516129032	1.007843137255	1.000488400488	
6	1.031746031746	1.001955034213	1.000030518044	
7	1.015748031496	1.000488400488	1.000001907350	1.000000000466
8	1.007843137255	1.000122077764	1.000000119209	1.000000000002
9	1.003913894325	1.000030518044	1.000000007451	1.000000000000
10	1.001955034213	1.000007629424	1.000000000466	1.000000000000
11	1.000977039570			
12	1.000488400488			
13	1.000244170431			
14	1.000122077764			
15	1.000061037019			
16	1.000030518044			
17	1.000015258905			
18	1.000007629424			
19	1.000003814705			
20	1.000001907350			

<u>PS</u>: $3/(5 - 4\cos\theta) =$

$$\ldots + \frac{1}{2}e^{-i\theta} + 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \frac{1}{8}e^{3i\theta} + \ldots$$

4 $\boxed{\displaystyle\int_0^{\pi/2} \frac{3}{5 - 4\cos\theta}\, d\theta}$

$$= \frac{\pi}{2} + \tan^{-1}\left(\frac{4}{3}\right) \simeq 2.498092$$

via just 1, 2, 3 or 4 trapez. steps, followed by a lot of EM "correcting" via odd derivs. at $\pi/2$, or in other words

	# of Trap. steps			
EM terms	n=1	n=2	n=3	n=4
0	2.827433	2.498734	2.488798	2.492063
1	2.926129	2.523408	2.499764	2.498232
2	2.914603	2.522688	2.499622	2.498187
3	2.917405	2.522732	2.499626	2.498187
4	2.918150	2.522734	2.499626	2.498187
5	2.912865	2.522729	2.499626	2.498187
6	2.926829	2.522733	2.499626	2.498187
7	2.905309	2.522731	2.499626	2.498187
8	2.828881	2.522730	2.499626	2.498187
9	3.975552	2.522735	2.499626	2.498187
10	-4.037105	2.522727	2.499626	2.498187
11	18.109478	2.522732	2.499626	2.498187
12	463.283258	2.522759	2.499626	2.498187

$$\int \ldots = \text{TRAPEZ} + \frac{1}{6}\frac{h^2}{2!}\left(1 - \frac{3}{4} + \frac{5}{4^2} - \frac{7}{4^3} + \ldots\right) + \frac{1}{30}\frac{h^4}{4!}\left(\ldots\right) + \ldots$$

5 $\boxed{\displaystyle\int_0^{\pi/2} \sin\theta\, d\theta}$ $= 1$,

likewise via just <u>one</u> step of length $h = \pi/2$, followed this time via lots of EM repairs referring to the lower limit, or

EM terms	approx. $\int$	|error|
0	.785398163397448	.214601836602552
1	.991014921753477	.008985078246523
2	.999470572016845	.000529427983155
3	.999967321558830	.000032678441170
4	.999997963567991	.000002036432009
5	.999999872813625	.000000127186375
6	.999999992052251	.000000007947749
7	.999999999503287	.000000000496713
8	.999999999968956	.000000000031044
9	.999999999998060	.000000000001940
10	.999999999999879	.000000000000121
11	.999999999999993	.000000000000008
12	1.000000000000000	.000000000000000

$$\int \ldots = \frac{\pi}{4} + \frac{1}{6}\frac{h^2}{2!} + \frac{1}{30}\frac{h^4}{4!} + \frac{1}{42}\frac{h^6}{6!} + \frac{1}{30}\frac{h^8}{8!} + \frac{5}{66}\frac{h^{10}}{10!} + \ldots$$

6 Preceding <u>sine integral</u> again, now (a) via Aitken/Shanks
(b) via Richardson

(a)

n	T_n	S^1	S^2	S^3
1	.785398163397			
2	.948059448969			
4	.987115800973	.999456721257		
8	.996785171886	.999966741765		
16	.999196680485	.999997932075	.999999963768	
32	.999799194320	.999999870921	.999999999432	
64	.999949800092	.999999991935	.999999999991	1.000000000000
128	.999987450118	.999999999496	1.000000000000	1.000000000000

(b)

n	T_n	R^1	R^2	R^3
1	.785398163397			
2	.948059448969	1.002279877492		
4	.987115800973	1.000134584974	.999991565473	
8	.996785171886	1.000008295524	.999999876227	1.000000008144
16	.999196680485	1.000000516685	.999999998095	1.000000000030
32	.999799194320	1.000000032265	.999999999970	1.000000000000
64	.999949800092	1.000000002016	1.000000000000	1.000000000000

A.T.