Course 18.327 and 1.130
Wavelets and Filter Banks

Accuracy of wavelet approximations (Condition A); vanishing moments; polynomial cancellation in filter banks

Accuracy of Wavelet Approximations

Biorthogonal case:

$$\tilde{\phi}(t) = 2\sum_n h_0[-n]\tilde{\phi}(2t - n) \qquad \phi(t) = 2\sum_n f_0[n]\phi(2t - n)$$

$$\tilde{w}(t) = 2\sum_n h_1[- n]\tilde{\phi}(2t - n) \qquad w(t) = 2\sum_n f_1[n]\phi(2t - n)$$

Biorthogonality means

$$\int_{-\infty}^{\infty} \phi(t)\, \tilde{\phi}(t - n)dt = \delta[n] \qquad \int_{-\infty}^{\infty} w(t)\, \tilde{w}(t - n)dt = \delta[n]$$

$$\int_{-\infty}^{\infty} \phi(t)\, \tilde{w}(t - n)dt = 0 \qquad \int_{-\infty}^{\infty} w(t)\, \tilde{\phi}(t - n)dt = 0$$

Suppose that $F_0(z)$ has p zeros at z = -1

$H_1(z) = F_0(-z) \quad \rightarrow$ p zeros at z = 1

i.e.

$$\frac{\partial^\ell}{\partial z^\ell} H_1(z) \Big|_{z=1} = 0 \quad \text{for } \ell = 0, 1, 2, \ldots, p-1$$

$$H_1(z) \Big|_{z=1} = \sum_n h_1[n] z^{-n} \Big|_{z=1} = \sum_n h_1[n]$$

$$\frac{\partial}{\partial z} H_1(z) \Big|_{z=1} = \sum_n (-n) h_1[n] z^{-n-1} \Big|_{z=1} = -\sum_n n\, h_1[n]$$

$$\frac{\partial^2}{\partial z^2} H_1(z) \Big|_{z=1} = \sum_n (-n)(-n-1) h_1[n] z^{-n-2} \Big|_{z=1} = \sum_n n^2 h_1[n] + \sum_n n h_1[n]$$

etc.

So

$$\sum_n n^\ell\, h_1[n] = 0 \qquad \text{for } \ell = 0, 1, 2, \ldots, p-1$$

Consider the moments of the analysis wavelet:

$$\int_{-\infty}^{\infty} t^{\ell}\, \tilde{w}(t)dt = 2\sum_n h_1[-n] \int_{-\infty}^{\infty} t^{\ell}\, \tilde{\phi}(2t - n)\, dt$$

$$= 2\sum_n h_1[-n] \int_{-\infty}^{\infty} \left(\frac{\tau + n}{2}\right)^{\ell} \tilde{\phi}(\tau)d\tau/2$$

$$= \frac{1}{2^{\ell}} \sum_n h_1[-n] \int_{-\infty}^{\infty} \sum_{i=0}^{\ell} \binom{\ell}{i} \tau^{\ell - i}\, n^i\, \tilde{\phi}(\tau)d\tau$$

$$= \frac{1}{2^{\ell}} \sum_{i=0}^{\ell} \binom{\ell}{i} \underbrace{\left(\sum_n h_1[n]\, n^i\right)}_{= 0 \text{ if } 0 \le i < p} (-1)^i \int_{-\infty}^{\infty} \tau^{\ell - i}\, \tilde{\phi}(\tau)d\tau$$

So the analysis wavelet has p vanishing moments:

$$\boxed{\int_{-\infty}^{\infty} t^{\ell}\, \tilde{w}(t)dt = 0 \quad \text{for } \ell = 0, 1, 2, \ldots, p - 1}$$

What do vanishing moments mean?
Try expanding the polynomial $\sum_{\ell=0}^{p-1} \alpha_\ell t^\ell$ in a wavelet basis:

$$P(t) \equiv \sum_{\ell=0}^{p-1} \alpha_\ell t^\ell = \sum_k c_{0,k}\, \phi_{0,k}(t) + \sum_{j\geq 0} \sum_k d_{j,k} w_{j,k}(t)$$

Then

$$d_{j,k} = \int_{-\infty}^{\infty} P(t)\tilde{w}_{j,k}(t)\, dt = \sum_{\ell=0}^{p-1} \alpha_\ell 2^{j/2} \int_{-\infty}^{\infty} t^\ell \tilde{w}(2^j t - k)\, dt$$

$$= 0$$

i.e. **polynomials of degree p – 1 can be expressed as a linear combination of scaling functions:**

$$\sum_{\ell=0}^{p-1} \alpha_\ell t^\ell = \sum_k c_{0,k}\, \phi(t - k) \text{ for some } c_{0,k}$$

Example (orthogonal wavelets)

$$f(t) = t^{\ell}$$

$$f_0(t) = \sum_k a_0[k]\,\phi(t-k) \in V_0 \; ; \; a_0[k] = \int_{-\infty}^{\infty} t^{\ell}\phi(t-k)\,dt$$

$$g_0(t) = \sum_k b_0[k]\,w(t-k) \in W_0 \; ; \; b_0[k] = \int_{-\infty}^{\infty} t^{\ell}w(t-k)\,dt$$

Suppose that $\phi(t)$ comes from a spline of degree $p-1$

($h_0[n]$ has p zeros at π) with $p-1 \geq \ell$.

Then we can write

$$t^{\ell} = \sum_k a[k]\,\phi(t-k)$$

The expansion coefficients are easily found since

$\phi(t-k)$ are orthonormal:

$$a[k] = \int_{-\infty}^{\infty} t^{\ell}\,\phi(t-k)\,dt$$

Also, since V_j h W_j, we have

$$\int_{-\infty}^{\infty} t^{\ell}\, w(t-n)dt = \sum_k a[k] \int_{-\infty}^{\infty} \phi(t-k)\, w(t-n)dt = 0$$

$\rightarrow$ vanishing moment property

So we have

$$f_0(t) = t^{\ell}$$

$$g_0(t) = 0$$

$$V_{j+1} = V_j \oplus W_j \quad \Rightarrow \quad f_{j+1}(t) = f_j(t) + g_j(t)$$

$$f_1(t) = f_0(t) + g_0(t) = t^{\ell}$$

All $f_j(t)$ are the same as f(t)!

Polynomial Data

Suppose x[n] = 1 for n $\geq$ 0 (unit step).

$$X(z) = \sum_{n=0}^{\infty} z^{-n}$$

$$\frac{d}{dz} X(z) = -\sum_{n=0}^{\infty} nz^{-n-1}$$

$$\frac{d^2}{dz^2} X(z) = \sum_{n=0}^{\infty} n(n+1)z^{-n-2}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{d^k}{dz^k} X(z) = (-1)^k \sum_{n=0}^{\infty} n(n+1)\ldots(n+k-1)z^{-n-k}$$

But we know that

$$X(z) = \frac{1}{1 - z^{-1}} \quad ; \quad |z| > 1$$

So

$$\frac{d}{dz} X(z) = \frac{-z^{-2}}{(1 - z^{-1})^2}$$

$$\frac{d^2}{dz^2} X(z) = \frac{2z^{-3}}{(1 - z^{-1})^3}$$

$$\vdots \qquad \qquad \vdots$$

$$\frac{d^k}{dz^k} X(z) = \frac{(-1)^k k!\, z^{-k-1}}{(1 - z^{-1})^{k+1}}$$

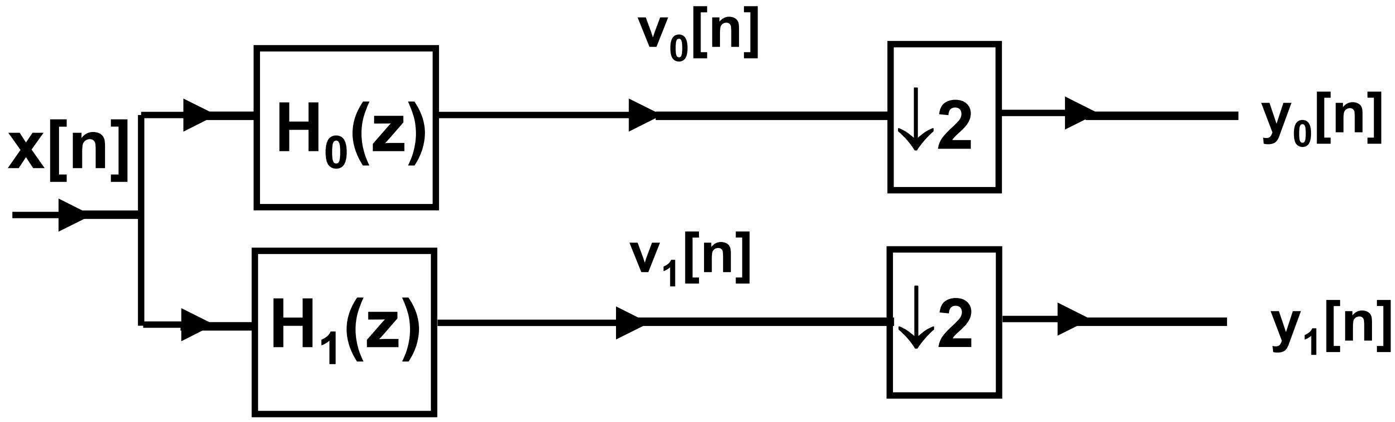
So if

$$x[n] = \underbrace{n(n+1)(n+2)\ldots(n+p-2)}_{\text{polynomial of degree } p-1} \quad ; \quad n \geq 0$$

then

$$X(z) = \frac{(p-1)!\, z^{-1}}{(1-z^{-1})^p} \quad ; \quad |z| > 1$$

Polynomial Data and Condition A_p



Consider Daubechies' filters

$$H_0(z) = (1 + z^{-1})^p Q(z) \qquad \text{p zeros at } z = -1$$

$$H_1(z) = -z^{-N} H_0(-z^{-1})$$

$$= (1 - z^{-1})^p R(z) \qquad ; \; R(z) = (-1)^{p-1} z^{-N+p} Q(-z^{-1})$$

Suppose that the input data is a polynomial of degree p-1:

$$x[n] = \sum_{k=0}^{p-1} a[k]\, S_p[n-k] \qquad \text{combination of shifts of } S_p[n]$$

where

$$S_p[n] = n(n+1)(n+2)\ldots(n+p-2) \qquad \text{for } n \geq 0$$

z-transform is

$$X(z) = \sum_{k=0}^{p-1} a[k] \frac{z^{-k}(p-1)!\, z^{-1}}{(1 - z^{-1})^p}$$

$$= \frac{(p-1)!\, z^{-1} A(z)}{(1 - z^{-1})^p} \quad ; \quad |z| > 1$$

Lowpass channel:

$$V_0(z) = H_0(z)\, X(z) = \frac{(p-1)!\, z^{-1} A(z)(1 + z^{-1})^p Q(z)}{(1 - z^{-1})^p}$$

So

$v_0[n]$ is a polynomial of degree p – 1.

$$y_0[n] = v_0[2n]$$

$\Rightarrow$ **$y_0[n]$ is a polynomial of degree p – 1.**

Highpass channel:

$$V_1(z) = H_1(z)\,X(z) = \frac{(p\text{-}1)!\, z^{-1} A(z)(1 - z^{-1})^p R(z)}{(1 - z^{-1})^p}$$

$$= (p\text{-}1)!\, z^{-1} A(z) R(z)$$

So $v_1[n]$ has finite length (even though $x[n]$ has infinite length.)

$$y_1[n] = v_1[2n]$$

$\Rightarrow$ $y_1[n]$ has finite length

i.e.

$y_1[n] = 0$, except for startup/boundary effects.