Incidence Problems in Plane and Higher Dimensions

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Abstract. This is a survey paper of recent results about the problem of counting number of incidences between points and curves in the plane. Also this discusses some results for this problem in higher dimensions, and tries to extend *crossing lemma* in higher dimensions.

Unit distance problem

Consider a set P of n points in the plane. The question is that, what is the maximum number of pairs of points in P that have unit distance. The history of this problem has been begun with Erdos's paper [9] in 1946. He found the bound $O(n^{3/2})$ for this problem. His proof was on the base of this fact that for two points with unit distance, there are at most two points with unit distance form both of them. In fact, if we consider the graph with n points as vertices and edges as pairs of unit distance points, this graph does not contain $K_{2,3}$ as a subgraph. Thus this bound is an immediate consequence of Turan theorem in extremal graph theory [10].

Let's look at the problem in another viewpoint. Consider a point in the plane. Set of all points that have unit distant from this point consist a circle with unit radius. So a point has unit distance with this point if it is on this circle. Therefore, in order to solve over problem, we can consider all the circles with radius one and center of n points and count the number of point-circle incidences.

Incidence problems

There are other problems, that are reduced to incidence problems. For example, consider these two problems:

What is the maximum number of unit area triangles determined by n points in the plane?

What is the maximum number of unit perimeter triangles determined by n points in the plane?

For the first problem, notice that in a triangle with area one and fixed two vertices the third vertex is on the union of two lines. Also for the second, in a unit perimeter triangle with fixed two vertices, the third vertex is on an ellipse. Thus, these problems reduced to point-line and point-ellipse incidence problems. It means, we have a set of points and a set of curves and want to count the number of pairs of point and curve in which the point is on the curve.

Point-line incidence

The first tight upper bound for these problems, was given by Szemeredi and Trotter in [17] for point-line incidence in plane. After that, in 1990, Clarkson, Edelsbrunner, Guibas, Sharir, and Welzl used another technique and proved that bound [6]. But in 1997, Szekely found a very simple proof for this theorem. His proof is based on a key lemma of Ajtai [2] and Leighton [11].

Lemma 1. *Let G be simple graph drawn in the plane, then either $e = O(v)$ or $Cr(G) = \Omega(e^3/v^2)$, where $e =\mid E(G) \mid$, $v =\mid V(G) \mid$ and $Cr(G)$ is the number of crossing of edges in the drawing.*

This lemma is called *crossing lemma* and there is a stronger version of that, due to Szekely [16]

Lemma 2. *Let G be multigraph drawn in the plane, with maximal edge-multiplicity M. Then either $e = O(Mv)$ or $Cr(G) = \Omega(e^3/Mv^2)$.*

Let's back to the Erdos's problem and use Szekely technique. We have n points and n circles around them with radius one, and want to count the number of point-circle incidences. Form a graph G with n points as vertices and pairs of consecutive points on circles as edges. In this graph number of edges is equal to I, number of incidences, and the multiplicity of any edge is at most two. Also the number of crosses is at most $2n(n-1)/2 < n^2$. So we have either $I = O(n)$, or $n^2 > Cr(G) = \Omega(I^3/2n^2)$ and then the number of unit distances is $I = O(n^{4/3})$, which is the bound of Spencer, Szemeredi and Trotter [15] in 1984, and is better than Erdos's bound.

So using crossing lemma we could count the number of point-circle incidences in the special case that circles have unit radios. Using these technique we can count the number of incidences in other cases. For example consider the point-line incidence, we get the theorem of Szemeredi and Trotter [17].

Theorem 1. *Consider set of n points and l lines in the plane, then the number of point-line incidences is*

$$O(n^{2/3}l^{2/3} + n + l).$$

This theorem can be proved exactly by the same idea as before, Szekely's method. The only point is that, for any two points there is at most one line passes through them, so the graph associated to this problem is simple and we can use the first version of crossing lemma. This point leads us to prove a more general theorem. In fact, instead of lines we can consider set of pseudo-lines, which is set of curves in the plane such that intersection of any two curves is at most one point. In this case we get the same bound as above. Notice that, this upper bound is tight. For example see Elekes in [8].

Point-circle incidence

Let's consider point-circle incidence in general case and apply this idea. Since for every two points there are infinity many circles pass through them we have not any bound on the edge-multiplicity of the graph. So we can't use Szekely's method directly. Now the idea is that, we can cut circles into some arcs such that any two arcs cross at most once. In fact Tamaki and Tokuyama in [18] show that any set of l circles can be cut into $O(l^{5/3})$ pseudo-segments. Using this bound and Szekely's technique we find the upper bound $O(n^{2/3}l^{2/3} + n + l^{5/3})$ for the number of point-circle incidences. But this bound is not tight. Now, the best bound for point-circle incidence is $O(n^{2/3}l^{2/3} + n^{6/11}l^{9/11}\kappa(n^3/l) + n + l)$, where $\kappa(n) = (\log n)^{O(\alpha^2(n))}$, and where $\alpha(n)$ is the inverse of Ackermann function [1], [3]. This bound comes from improving the bound of Tamaki and Tokuyama for the number of cuts of circles into pseudo-segments.

Curves with k degree of freedom

Now, consider ellipses instead of circles or curves of degree three or more. Two ellipses can have four points in their intersection, so the problem for ellipses is more complicated. In order to solve these problem, Pach and sharir stated a definition for curves in [13]:

Definition. Let C be a given class of simple curves in the plane. We say that C has k degree of freedom and multiplicity-type s if

(i) for any k points there are at most s curves of C passing through all of them,

(ii) any pair of curves from C intersect in at most s points.

Their idea is that for constructing a graph we should not just considering consecutive pairs of points on curves, but add some edges between every two points that, there are at most $k-1$ points between them on the curve. By this construction we can consider this fact that every k points specify finitely many curves, not just two points. But in this case, we don't have any bound on M, the multiplicity-edge of graph. The technique is the same as that for circles. We can partition vertices and using crossing lemma in each part after deleting some edges in order to have a bound on multiplicity in each part. By this technique they could prove the following theorem in [13].

Theorem 2. *let P be a set of n points and C be a set of l simple curves all lying in the plane. If C has k degrees of freedom and multiplicity-type s, then the number of point-curve incidences is*

$$O(n^{k/(2k-1)}l^{(2k-2)/(2k-1)} + n + l).$$

Notice that this upper bound in not tight. For example set of circles have 3 degree of freedom. So we get the bound $O(n^{3/5}l^{4/5} + n + l)$ which is not tight.

Incidences in higher dimensions

It seems that by theorem 2 we have an *admissible* bound for incidence problems in plane. But a natural extension of this problem is the incidence problems in higher dimensions. For example incidences between hyperplanes or spheres and points in $\mathbb{R}^d$, also point-curve incidences in higher dimensions.

point-hyperplane incidence

The question is that, what's the maximum number of incidences for a set of n points and l hyperplanes. Notice that, without any restriction on points and hyperplanes we can have nl incidences, because we can consider n points on a line such that all hyperplanes pass through that line. Now in order to have some restrictions, we can assume that no three points are collinear, or no three hyperplanes have a line in their intersection. In fact, these conditions are some assumption in order to not have any $K_{r,r}$ as a subgraph of incidence graph for large numbers r. Brass and Knauer [5] show that the number of incidences between n points and l hyperplanes in $\mathbb{R}^d$ is

$$O((n+l)\log(n+l) + n^{d/(d+1)} l^{d/(d+1)} \log(nl))$$

by the condition that their incidence graph doesn't contain $K_{r,r}$ for a fixed r.

Point and unit spheres incidence in $\mathbb{R}^3$

There's a technique for solving incidence problems that is partition. For example, we can prove the theorem of Szemeredi and Trotter by this method. Let's solve the problem of incidence for unit spheres by this technique. This problem is related to the problem of maximum number of unit distances in an arrangement of a point set in three dimension. We know that for three points in the space there are at most two unit spheres passing through them. On the other word, the incidence graph does not contain $K_{3,3}$, so by Turan's theorem [10] number of incidences is $O(nl^{2/3} + l)$. But this bound is not tight, and the idea is that we can partition the space into some parts and use this bound in each partition. In fact, Clarkson et al. in [6] show that we can partition the space into $O(r^3\beta(r))$ cells such that, each cell crosses at most l/r spheres, where $\beta(r) = 2^{O(\alpha^2(r))}$. Now apply the above bound in any partition. If there are n_i points in the $i-th$ partition, then the number of incidences in this part is $O(n_i(l/r)^{2/3} + l/r)$, and sum over all cells, we get the number of incidences is $O(n(l/r)^{2/3} + lr^2\beta(r))$. Now choose $r = n^{3/8}/l^{1/8}$ when $l^{1/3} \leq n \leq l^3$, we find the bound $O(n^{3/4}l^{3/4}\beta(n+l) + n + l)$. Also, if n is not in that range, one can easily check that this bound works.

Point-cylinder incidence

We have shown that the maximum number of unit area triangles with vertices in a set of n points in the plane is related to point-line problem. Now consider this problem in three dimension. For a unit area triangle in space such that two vertices of that, are fixed, the third vertex can be on a cylinder. Thus this problem in three dimensions is related to point-cylinder incidence problem. But notice that, this problem without any restriction is trivial, same as point-plane incidence, all cylinders can have a line in their intersection. One restriction is to assume that, the axis of any cylinder passes through origin. In this case, since every cylinder is set of points satisfying a degree two polynomial, every three cylinder have at most eight points in their intersection. It means that the incidence graph for cylinders contains no $K_{9,3}$, so by Turan's theorem [10], for n points and l cylinders we have at most $O(nl^{8/9} + l)$ or $O(n^{2/3}l + n)$ incidences. It seems that these bounds can be better by the partition idea.

point-line incidence in higher dimensions

Consider the point-line incidence in three dimension. Since the bound of Szemeredi and Trotter [17] for point-line incidence in plane is tight, the bound for maximum number of point-line incidences in three space, is at list $O(n^{2/3}l^{2/3} + n + l)$. On the other hand, for every set of points and lines in space, we can project them in a general plane. In this case, we have the same number of points, lines and incidences, thus we have exactly the same bound for point-line incidences in space as in plane.

Sharir and Welzl mentioned this point in [14], and tried to set up this problem in another way. They used the concept of *joint*, considered a weight for each joint, and proved an upper bound for the sum of weights. Let P be a set of n points and L set of l lines in space. For a point $p \in P$ define L_p, set of lines in L pass through p. We call p a joint of L, if L_p contains at least three non-coplanar lines, and let J_L the set of all joints. Also let c_p, denote the minimum number of planes that contain all lines in L_p. In fact, a point p is a joint iff $c_p \geq 2$. Now define $I_c(P, L)$ to be the sum of c_p's over all points $p \in P$. Sharir and Welzl proved that the incidence number between J_L and L is $O(l^{5/3})$, and used this fact to show that $I_c(P, L) = O(n^{4/7}l^{5/7} + n + l)$.

Another way to set up this problem is to suppose each line forms a fixed angle with the xy-plane. In this case, they proved number of incidences is

$$O(min\{n^{3/4}l^{1/2}\kappa(n), n^{4/7}l^{5/7}\} + n + l).$$

Notice that both of these bounds are smaller than the bound of Szemeredi and Trotter.

Point-circle incidence in higher dimensions

Aronov, Koltun and Sharir in [4], stated the problem of point-circle incidence in three and higher dimensions. First of all, they proved the number of incidences between n points and l circles in $\mathbb{R}^3$ is

$$O(n^{2/3}l^{2/3} + l^{3/2}\kappa(l) + n).$$

The idea of their proof is that, for a circle c that has many intersections with other circles, we can consider some spheres pass through c and cover all those circles that have intersection with it. Now project each of these spheres onto a plane by a general point on that sphere. We get a set of points and circles in the plane and we can apply the bound of point-circle incidence for each of these planes. Sum over all these spheres, we get the upper bound $O(n^{2/3}l^{2/3} + l^{3/2}\kappa(l) + n)$ for the number of incidences. This bound is optimal when $n \geq l^{5/4}\kappa^{3/2}(l)$. For smaller values of n, they apply another method. In fact, they used the duality technique. Suppose no pair of circles are coplanar, and apply the standard duality transform that maps each point to a plane, and each plane to a point. This transform preserve incidence. Now for each circle consider its plane, and associate to each circle, the dual of this plane. So we receive a point-plane incidence problem in dual space. Applying the partition method, we can cut the space into $O(r^3)$ simplices, such that each simplex is intersected by at most n/r planes of the dual of points. Now apply the previous bound in each simplex we get the bound $O(n^{6/11}l^{9/11}\kappa(n^3/l) + n^{2/3}l^{2/3} + n + l)$. They showed that, both of these bound works for circles in $\mathbb{R}^d$, for any $d \geq 3$. Also they found a bound for number of incidences between a set of n points and l convex curves that belongs to a two dimensional plane.

Theorem 3. *Let a collection of n points and l convex plane curves in $\mathbb{R}^d$, such that, no two of which lie in a common 2-plane. Then for any $d \geq 3$, the number of point-curve incidences is*

$$O(n^{4/7}l^{17/21} + n^{2/3}l^{2/3} + n + l).$$

Extremal problems for geometric hypergraphs

One of the most useful methods in solving incidence problems is *crossing lemma*, that can be used for incidence problems in plane, but we don't have such lemma in higher dimensions. It seems that such lemmas can be useful for proving some results of incidence problems in higher dimensions. Dey and Pach, in [7], have defined geometric hypergraphs, and tried to generalize *crossing lemma*.

A *d-dimensional geometric r-hypergraph,* H_r^d is a pair (V, E), where V is a set of points in general position in $\mathbb{R}^d$, and E is a set of closed $(r-1)$-dimensional simplices induced by some r-tuples of V. The sets V and E are called the vertex set and edge set of H_r^d, respectively.

Now notice that the notion of crossing is not clear in higher dimensions. For example if two edge of two triangles in space cross, are the triangles cross or not? In order to clarify the terminology, they stated this definition:

k simplices are said to have a *nontrivial intersection*, if their relative interiors have a point in common. If, in addition, the k simplices are vertex disjoint, then they are said to *cross*. Notice that, if every pair of k simplices, has a nontrivial intersection, it does not imply that all of them do. They proved the following theorems:

Theorem 4. *Let E be any set of d-dimensional simplices induced by an n-element point set $V \subseteq \mathbb{R}^d$. If E has no two crossing elements, then $\mid E \mid= \Theta(n^d)$.*

Theorem 5. *Let E be a family of $(d-1)$-dimensional simplices induced by an n-element point set $V \subseteq \mathbb{R}^d$, where $d, k > 1$. If E has no k pairwise crossing members, then $\mid E \mid= O(n^{d-(1/d)^{k-2}})$.*

Other than these bounds they could found some upper bounds for the number of crossing edges in any d-dimensional geometric r-hypergraph.

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