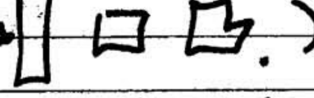
Pak
12/14/05

Introduction to tilings

Region Γ , tiles (eg .)

Question 1. Given $T = \{\tau_1, \tau_2, ..\}$ set of tiles, is Γ tileable

Q2 How many ways

etc.

① First piece of trivia

Dominoes: # ways to tile $2\times n$ is Fibonacci

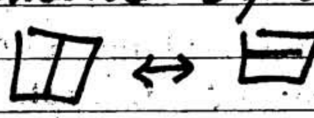
② Checkerboard (nec not suff)

③ dom tilings $\leftrightarrow$ perfect matching of dual, which can be done in poly time $\Rightarrow$ $\exists$ tiling? in poly time (bip $\Rightarrow$ $O(|E||V|)$?)

Thm (stated + vaguely proved by Thurston)

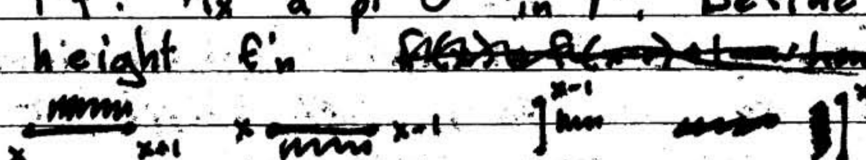
Γ simp. conn $\Rightarrow \exists$ lin time alg to determine tileability of Γ by dominoes (lin w.r.t. area)

Thm (Thurston)

If Γ is simply connected, every two domino tilings are connected by a sequence of 2×2 moves 

Pf: Fix a pt O in Γ. Define $f: \partial\Gamma \to \mathbb{Z}$

height f'n ~~[illegible]~~



(picture underlying chessboard + look at color of each square)

Claim: Γ s.c., tileable $\Rightarrow$ f well defined

Pf By induction, Remove a domino on Γ (not just $\partial\Gamma$), check around that domino, ✓

Lemma: $\Gamma \overset{\subseteq \mathbb{R}^2}{\text{s.c.}}$, $\Gamma = \bigsqcup^{K} R_i$, R_i s.c., $\Rightarrow$ $\exists\, i \in [K]$ s.t. $\Gamma \overset{i \le l}{\setminus} R_i$ is s.c.

(use lemma to show you can remove domino in pf)

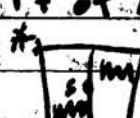
(Exc: false in $\mathbb{R}^3$)

given height f'n's can reconstruct the domino tiling by connecting pts x $x+1$ v

Define poset P_Γ on all domino tilings of Γ / $f < f'$ if $\forall$ s pt in Γ $f(s) < f'(s)$ height f'ns

Lemma $\exists!$ max, $\exists!$ min $\in P_\Gamma$

Pf of Lemma: Lemma f local min $\Rightarrow$ max $f(s) \in \partial\Gamma$

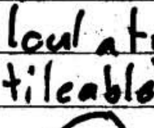


but

Pf: Suppose not. then $s \notin \Gamma$ (since s is max) must have *

flipping that square gives less $\lightning$

This lemma of lemma allows us to prove the whole theorem.

Algorithm: Calculate ht f'n on $\partial\Gamma$ (well defined if tileable). Find max s, must have domino , remove it, repeat.

This gives local (hence global) min'm / max

Furthermore, can keep 2×2 flipping to decrease ht f'n until reach global min