Pak
9/16/05

In general, $\ge$ // Thm (E-L) Let $\mathcal{F}$ be K-uniform, intersecting, and $|\mathcal{F}| > K^K$. Then $\mathcal{F}$ is 2-colorable

Pf: By (contrapositive) contradiction, $\mathcal{F}$ not 2-col'ble $\Rightarrow |\mathcal{F}| \le K^K$

Let $d(B) = \# S_i \in \mathcal{F}$ s.t. $B \subset S_i$

Claim: $\exists$ a sequence $x_1, \ldots x_K \in [n]$ s.t.

$d(B_i) \ge \frac{|\mathcal{F}|}{K^i}$ $\quad B_i = \{x_1 \ldots x_i\}$

Claim $\Rightarrow$ thm: $1 \ge d(B_K) \ge \frac{|\mathcal{F}|}{K^K}$ ✓

Pf of claim by induction on i:

Base: $i=1$ pigeonhole ✓

(look at S_1 + what's in itersection w/ other stuff, one elt is in at least $\frac{|\mathcal{F}|-1}{K}$ of 'em) ✓

Step of Induction: Suppose B_i $i<K$ already chosen.

If $B_i \cap S_j \ne \phi$ $\forall S_j \in \mathcal{F}$, can 2-color Σ

So let $x_{i+1} \in S_j$ be the one which belongs to largest # of subsets containing B_i.

This is $\ge d(B_i)/K$ by sim. argument to base ✓

"OKay, so, I'm going to move to colorings of graphs."

Notation: Let $d(G) = \max_{v \in G} d(v)$

$\chi(G) =$ usu.

Proposition

~~Thm~~ $d(G)=K \Rightarrow \chi(G) \le K+1$ ✓

Prop'n Connected $d(G)=K$, $\exists\, x \in V(G)$ s.t. $d(v) \le K-1 \Rightarrow \chi(G) \le K$

Pf: Let small deg vertex be root, orient edges acyclicly

sort out vertices by distance from x, note all

edges are w/in same dist or from ~~closer~~ far to close

$x \quad y_1 \quad \dots \quad y_{n-1}$ Run greedy algorithm starting

$y_{n-1} \quad \dots \quad x$. Note that always have

one uncolored neighbor until x + $d(x)=K-1$ ✓

(similar in Bollobás) (aha!)

(This is a real) Theorem (Brooks 1949)

$d(G) \le K$ and $G \ne K_{K+1}$ and $K \ge 3$.

Then $\chi(G) \le K$. (G connected)

Pf: Need only prove for G being K-regular, as o/w above.

(Def'n: G is r-connected if $\forall$ $r-1$ vertices $y_1 \dots y_{r-1}$

$G - y_1 - \dots y_{r-1}$ is connected)

Okay, for G 2-conn, pinch vertex ✓

G 3-conn. Take x_n, let $x_1, x_2 \in N(x_n)$

$\{x_1, x_2\} \notin E(G)$. Look at $G - x_1, x_2$. G 3-conn

$\Rightarrow$ still connected. Construct sequence w/

x_n as a root, forward neighbor rule, x_1 and x_2

1st 2 vertices + color them the same

color $\Rightarrow$ ✓ cool.