LECTURE 3
Matroids and geometric lattices

3.1. Matroids

A matroid is an abstraction of a set of vectors in a vector space (for us, the normals to the hyperplanes in an arrangement). Many basic facts about arrangements (especially linear arrangements) and their intersection posets are best understood from the more general viewpoint of matroid theory. There are many equivalent ways to define matroids. We will define them in terms of independent sets, which are an abstraction of linearly independent sets. For any set S we write

$$2^S = \{T : T \subseteq S\}.$$

Definition 3.8. A (finite) *matroid* is a pair $M = (S, \mathcal{I})$, where S is a finite set and $\mathcal{I}$ is a collection of subsets of S, satisfying the following axioms:

(1) $\mathcal{I}$ is a nonempty (abstract) *simplicial complex*, i.e., $\mathcal{I} \neq \emptyset$, and if $J \in \mathcal{I}$ and $I \subset J$, then $I \in \mathcal{I}$.
(2) For all $T \subseteq S$, the maximal elements of $\mathcal{I} \cap 2^T$ have the same cardinality. In the language of simplicial complexes, every induced subcomplex of $\mathcal{I}$ is *pure*.

The elements of $\mathcal{I}$ are called *independent sets*. All matroids considered here will be assumed to be finite. By standard abuse of notation, if $M = (S, \mathcal{I})$ then we write $x \in M$ to mean $x \in S$. The archetypal example of a matroid is a finite subset S of a vector space, where independence means linear independence. A closely related matroid consists of a finite subset S of an affine space, where independence now means affine independence.

It should be clear what is meant for two matroids $M = (S, \mathcal{I})$ and $M' = (S', \mathcal{I}')$ to be *isomorphic*, viz., there exists a bijection $f : S \to S'$ such that $\{x_1, \dots, x_j\} \in \mathcal{I}$ if and only if $\{f(x_1), \dots, f(x_j)\} \in \mathcal{I}'$. Let M be a matroid and S a set of points in $\mathbb{R}^n$, regarded as a matroid with independence meaning affine independence. If M and S are isomorphic matroids, then S is called an *affine diagram* of M. (Not all matroids have affine diagrams.)

Example 3.7. (a) Regard the configuration in Figure 1 as a set of five points in the two-dimensional affine space $\mathbb{R}^2$. These five points thus define the affine diagram of a matroid M. The lines indicate that the points 1,2,3 and 3,4,5 lie on straight

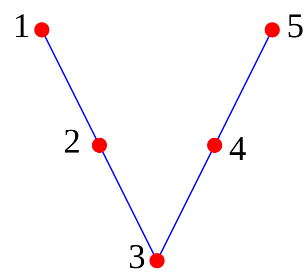


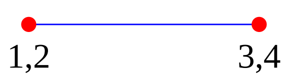
Figure 1. A five-point matroid in the affine space $\mathbb{R}^2$

lines. Hence the sets $\{1,2,3\}$ and $\{3,4,5\}$ are affinely dependent in $\mathbb{R}^2$ and therefore dependent (i.e., not independent) in M. The independent sets of M consist of all subsets of $[5]$ with at most two elements, together with all three-element subsets of $[5]$ except 123 and 345 (where 123 is short for $\{1,2,3\}$, etc.).

(b) Write $\mathcal{I} = \langle S_1, \dots, S_k \rangle$ for the simplicial complex $\mathcal{I}$ *generated* by $S_1, \dots, S_k$, i.e.,

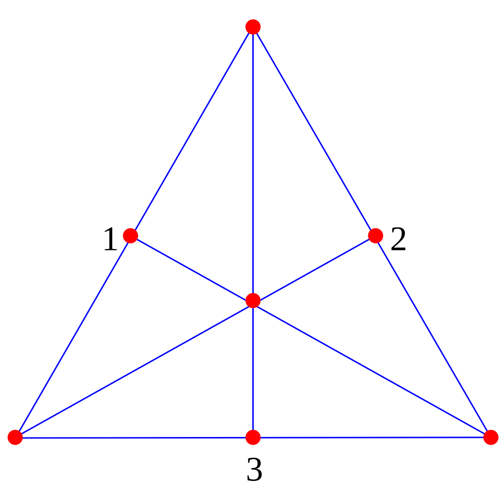
$$\begin{aligned} \langle S_1, \dots, S_k \rangle &= \{T : T \subseteq S_i \text{ for some } i\} \\ &= 2^{S_1} \cup \cdots \cup 2^{S_k}. \end{aligned}$$

Then $\mathcal{I} = \langle 13, 14, 23, 24 \rangle$ is the set of independent sets of a matroid M on $[4]$. This matroid is realized by a *multiset* of vectors in a vector space or affine space, e.g., by the points 1,1,2,2 in the affine space $\mathbb{R}$. The affine diagam of this matroid is given by

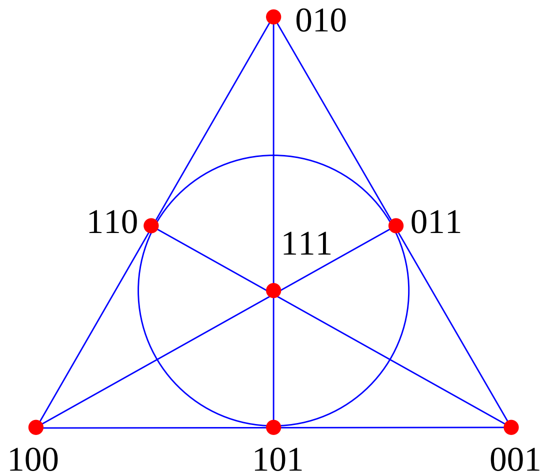


(c) Let $\mathcal{I} = \langle 12, 23, 34, 45, 15 \rangle$. Then $\mathcal{I}$ is *not* the set of independent sets of a matroid. For instance, the maximal elements of $\mathcal{I} \cap 2^{\{1,2,4\}}$ are 12 and 4, which do not have the same cardinality.

(d) The affine diagram below shows a seven point matroid.



If we further require the points labelled 1,2,3 to lie on a line (i.e., remove 123 from $\mathcal{I}$), we still have a matroid M, but not one that can be realized by real vectors. In fact, M is isomorphic to the set of nonzero vectors in the vector space $\mathbb{F}_2^3$, where $\mathbb{F}_2$ denotes the two-element field.



Let us now define a number of important terms associated to a matroid M. A *basis* of M is a maximal independent set. A *circuit* C is a minimal dependent set, i.e., C is not independent but becomes independent when we remove any point from it. For example, the circuits of the matroid of Figure 1 are 123, 345, and 1245.

If $M = (S, \mathcal{I})$ is a matroid and $T \subseteq S$ then define the *rank* $\mathrm{rk}(T)$ of T by

$$\mathrm{rk}(T) = \max\{\#I : I \in \mathcal{I} \text{ and } I \subseteq T\}.$$

In particular, $\mathrm{rk}(\emptyset) = 0$. We define the rank of the matroid M itself by $\mathrm{rk}(M) = \mathrm{rk}(S)$. A *k-flat* is a maximal subset of rank k. For instance, if M is an affine matroid, i.e., if S is a subset of an affine space and independence in M is given by affine independence, then the flats of M are just the intersections of S with affine subspaces. Note that if F and F' are flats of a matroid M, then so is $F \cap F'$ (see Exercise 2). Since the intersection of flats is a flat, we can define the *closure* $\overline{T}$ of a subset $T \subseteq S$ to be the smallest flat containing T, i.e.,

$$\overline{T} = \bigcap_{\text{flats } F \supseteq T} F.$$

This closure operator has a number of nice properties, such as $\overline{\overline{T}} = \overline{T}$ and $T' \subseteq T \Rightarrow \overline{T}' \subseteq \overline{T}$.

3.2. The lattice of flats and geometric lattices

For a matroid M define $L(M)$ to be the poset of flats of M, ordered by inclusion. Since the intersection of flats is a flat, $L(M)$ is a meet-semilattice; and since $L(M)$ has a top element S, it follows from Lemma 2.3 that $L(M)$ is a lattice, which we call the *lattice of flats* of M. Note that $L(M)$ has a unique minimal element $\hat{0}$, viz., $\bar{\emptyset}$ or equivalently, the intersection of all flats. It is easy to see that $L(M)$ is graded by rank, i.e., every maximal chain of $L(M)$ has length $m = \mathrm{rk}(M)$. Thus if $x \lessdot y$ in

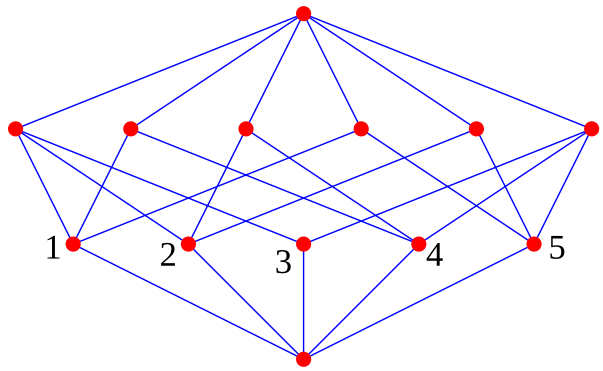


Figure 2. The lattice of flats of the matroid of Figure 1

$L(M)$ then $\mathrm{rk}(y) = 1 + \mathrm{rk}(x)$. We now define the *characteristic polynomial* $\chi_M(t)$, in analogy to the definition (3) of $\chi_{\mathcal{A}}(t)$, by

$$\chi_M(t) = \sum_{x \in L(M)} \mu(\hat{0}, x) t^{m - \mathrm{rk}(x)}, \tag{22}$$

where μ denotes the Möbius function of $L(M)$ and $m = \mathrm{rk}(M)$. Figure 2 shows the lattice of flats of the matroid M of Figure 1. From this figure we see easily that

$$\chi_M(t) = t^3 - 5t^2 + 8t - 4.$$

Let M be a matroid and $x \in M$. If the set $\{x\}$ is dependent (i.e., if $\mathrm{rk}(\{x\}) = 0$) then we call x a *loop*. Thus $\bar{\emptyset}$ is just the set of loops of M. Suppose that $x, y \in M$, neither x nor y are loops, and $\mathrm{rk}(\{x, y\}) = 1$. We then call x and y *parallel* points. A matroid is *simple* if it has no loops or pairs of parallel points. It is clear that the following three conditions are equivalent:

- M is simple.
- $\bar{\emptyset} = \emptyset$ and $\bar{x} = x$ for all $x \in M$.
- $\mathrm{rk}(\{x, y\}) = 2$ for all points $x \neq y$ of M (assuming M has at least two points).

For any matroid M and $x, y \in M$, define $x \sim y$ if $\bar{x} = \bar{y}$. It is easy to see that $\sim$ is an equivalence relation. Let

$$\widehat{M} = \{\bar{x} \,:\, x \in M,\ x \notin \bar{\emptyset}\}, \tag{23}$$

with an obvious definition of independence, i.e.,

$$\{\bar{x}_1, \ldots, \bar{x}_k\} \in \mathcal{I}(\widehat{M}) \Leftrightarrow \{x_1, \ldots, x_k\} \in \mathcal{I}(M).$$

Then $\widehat{M}$ is simple, and $L(M) \cong L(\widehat{M})$. Thus insofar as intersection lattices $L(M)$ are concerned, we may assume that M is simple. (Readers familiar with point set topology will recognize the similarity between the conditions for a matroid to be simple and for a topological space to be T_0.)

Example 3.8. Let S be any finite set and V a vector space. If $f : S \to V$, then define a matroid M_f on S by the condition that given $I \subseteq S$,

$$I \in \mathcal{I}(M) \Leftrightarrow \{f(x) \,:\, x \in I\} \text{ is linearly independent.}$$

Then a loop is any element x satisfying $f(x) = 0$, and $x \sim y$ if and only if $f(x)$ is a nonzero scalar multiple of $f(y)$.

NOTE. If $M = (S, \mathcal{I})$ is simple, then $L(M)$ determines M. For we can identify S with the set of atoms of $L(M)$, and we have

$$\{x_1, \ldots, x_k\} \in \mathcal{I} \;\Leftrightarrow\; \mathrm{rk}(x_1 \vee \cdots \vee x_k) = k \text{ in } L(M).$$

See the proof of Theorem 3.8 for further details.

We now come to the primary connection between hyperplane arrangements and matroid theory. If H is a hyperplane, write $\mathfrak{n}_H$ for some (nonzero) normal vector to H.

Proposition 3.6. *Let $\mathcal{A}$ be a* central *arrangement in the vector space V. Define a matroid $M = M_{\mathcal{A}}$ on $\mathcal{A}$ by letting $\mathcal{B} \in \mathcal{I}(M)$ if $\mathcal{B}$ is linearly independent (i.e., $\{\mathfrak{n}_H : H \in \mathcal{B}\}$ is linearly independent). Then M is simple and $L(M) \cong L(\mathcal{A})$.*

Proof. M has no loops, since every $H \in \mathcal{A}$ has a nonzero normal. Two distinct nonparallel hyperplanes have linearly independent normals, so the points of M are closed. Hence M is simple.

Let $\mathcal{B}, \mathcal{B}' \subseteq \mathcal{A}$, and set

$$X = \bigcap_{H \in \mathcal{B}} H = X_{\mathcal{B}}, \quad X' = \bigcap_{H \in \mathcal{B}'} H = X_{\mathcal{B}'}.$$

Then $X = X'$ if and only if

$$\mathrm{span}\{\mathfrak{n}_H : H \in \mathcal{B}\} = \mathrm{span}\{\mathfrak{n}_H : H \in \mathcal{B}'\}.$$

Now the closure relation in M is given by

$$\overline{\mathcal{B}} = \{H' \in \mathcal{A} : \mathfrak{n}_{H'} \in \mathrm{span}\{\mathfrak{n}_H : H \in \mathcal{B}\}\}.$$

Hence $X = X'$ if and only if $\overline{B} = \overline{B}'$, so $L(M) \cong L(\mathcal{A})$. □

It follows that for a central arrangement $\mathcal{A}$, $L(\mathcal{A})$ depends only on the *matroidal* structure of $\mathcal{A}$, i.e., which subsets of hyperplanes are linearly independent. Thus the matroid $M_{\mathcal{A}}$ encapsulates the essential information about $\mathcal{A}$ needed to define $L(\mathcal{A})$.

Our next goal is to characterize those lattices L which have the form $L(M)$ for some matroid M.

Proposition 3.7. *Let L be a finite graded lattice. The following two conditions are equivalent.*

(1) *For all $x, y \in L$, we have $\mathrm{rk}(x) + \mathrm{rk}(y) \geq \mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y)$.*
(2) *If x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y.*

Proof. Assume (1). Let $x, y \gtrdot x \wedge y$, so $\mathrm{rk}(x) = \mathrm{rk}(y) = \mathrm{rk}(x \wedge y) + 1$ and $\mathrm{rk}(x \vee y) > \mathrm{rk}(x) = \mathrm{rk}(y)$. By (1),

$$\begin{aligned} \mathrm{rk}(x) + \mathrm{rk}(y) &\geq (\mathrm{rk}(x) - 1) + \mathrm{rk}(x \vee y) \\ \Rightarrow \mathrm{rk}(y) &\geq \mathrm{rk}(x \vee y) - 1 \\ \Rightarrow x \vee y &\gtrdot x. \end{aligned}$$

Similarly $x \vee y \gtrdot y$, proving (2).

For (2)⇒(1), see [**18**, Prop. 3.3.2]. □

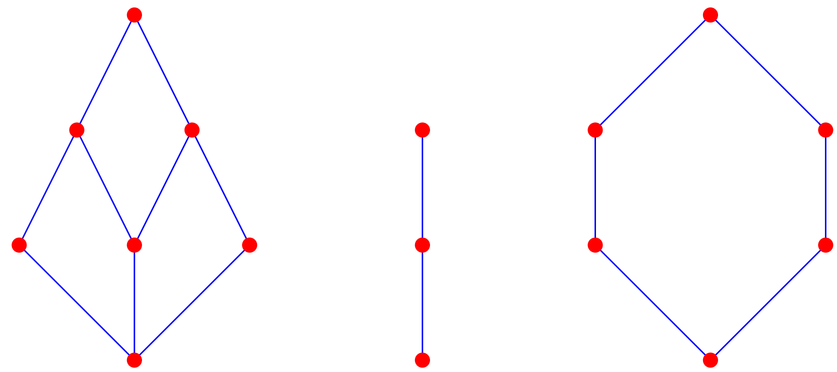


Figure 3. Three nongeometric lattices

Definition 3.9. A finite lattice L satisfying condition (1) or (2) above is called *(upper) semimodular*. A finite lattice L is *atomic* if every $x \in L$ is a join of atoms (where we regard $\hat{0}$ as an empty join of atoms). Equivalently, if $x \in L$ is join-irreducible (i.e., covers a unique element), then x is an atom. Finally, a finite lattice is *geometric* if it is both semimodular and atomic.

To illustrate these definitions, Figure 3(a) shows an atomic lattice that is not semimodular, (b) shows a semimodular lattice that is not atomic, and (c) shows a graded lattice that is neither semimodular nor atomic.

We are now ready to characterize the lattice of flats of a matroid.

Theorem 3.8. *Let L be a finite lattice. The following two conditions are equivalent.*

1. *L is a geometric lattice.*
2. *$L \cong L(M)$ for some (simple) matroid M.*

Proof. Assume that L is geometric, and let A be the set of atoms of L. If $T \subseteq A$ then write $\bigvee T = \bigvee_{x \in T} x$, the join of all elements of T. Let

$$\mathcal{I} = \{I \subseteq A : \mathrm{rk}(\vee I) = \#I\}.$$

Note that by semimodularity, we have for any $S \subseteq A$ and $x \in A$ that $\mathrm{rk}((\bigvee S) \vee x) \leq \mathrm{rk}(\bigvee S) + 1$. (Hence in particular, $\mathrm{rk}(\bigvee S) \leq \#S$.) It follows that $\mathcal{I}$ is a simplicial complex. Let $S \subseteq A$, and let T, T' be maximal elements of $2^S \cap \mathcal{I}$. We need to show that $\#T = \#T'$.

Assume $\#T < \#T'$, say. If $y \in S$ then $y \leq \bigvee T'$, else $T'' = T' \cup y$ satisfies $\mathrm{rk}(\bigvee T'') = \#T''$, contradicting the maximality of T'. Since $\#T < \#T'$ and $T \subseteq S$, it follows that $\bigvee T < \bigvee T'$ [why?]. Since L is atomic, there exists $y \in S$ such that $y \in S$ but $y \not\leq \bigvee T$. But then $\mathrm{rk}(\bigvee(T \cup y)) = 1 + \#T$, contradicting the maximality of T. Hence $M = (A, \mathcal{I})$ is a matroid, and $L \cong L(M)$.

Conversely, given a matroid M, which we may assume is simple, we need to show that $L(M)$ is a geometric lattice. Clearly $L(M)$ is atomic, since every flat is the join of its elements. Let $S, T \subseteq M$. We will show that

$$\mathrm{rk}(S) + \mathrm{rk}(T) \geq \mathrm{rk}(S \cap T) + \mathrm{rk}(S \cup T). \tag{24}$$

Note that if S and T are *flats* (i.e., $S, T \in L(M)$) then $S \cap T = S \wedge T$ and $\mathrm{rk}(S \cup T) = \mathrm{rk}(S \vee T)$. Hence taking S and T to be flats in (24) shows that $L(M)$ is semimodular and thus geometric. Suppose (24) is false, so

$$\mathrm{rk}(S \cup T) > \mathrm{rk}(S) + \mathrm{rk}(T) - \mathrm{rk}(S \cap T).$$

Let B be a basis for $S \cup T$ extending a basis for $S \cup T$. Then either $\#(B \cap S) > \mathrm{rk}(S)$ or $\#(B \cap T) > \mathrm{rk}(T)$, a contradiction completing the proof. □

Note that by Proposition 3.6 and Theorem 3.8, any results we prove about geometric lattices hold a fortiori for the intersection lattice $L_{\mathcal{A}}$ of a central arrangement $\mathcal{A}$.

NOTE. If L is geometric and $x \leq y$ in L, then it is easy to show using semimodularity that the interval $[x, y]$ is also a geometric lattice. (See Exercise 3.) In general, however, an interval of an atomic lattice need not be atomic.

For noncentral arrangements $L(\mathcal{A})$ is not a lattice, but there is still a connection with geometric lattices. For a stronger statement, see Exercise 4.

Proposition 3.8. *Let $\mathcal{A}$ be an arrangement. Then every interval $[x, y]$ of $L(\mathcal{A})$ is a geometric lattice.*

Proof. By Exercise 3, it suffices to take $x = \hat{0}$. Now $[\hat{0}, y] \cong L(\mathcal{A}_y)$, where $\mathcal{A}_y$ is given by (6). Since $\mathcal{A}_y$ is a central arrangement, the proof follows from Proposition 3.6. □

The proof of our next result about geometric lattices will use a fundamental formula concerning Möbius functions known as *Weisner's theorem*. For a proof, see [**18**, Cor. 3.9.3] (where it is stated in dual form).

Theorem 3.9. *Let L be a finite lattice with at least two elements and with Möbius function μ. Let $\hat{0} \neq a \in L$. Then*

$$\sum_{x\,:\,x \vee a = \hat{1}} \mu(x) = 0. \tag{25}$$

Note that Theorem 3.9 gives a "shortening" of the recurrence (2) defining μ. Normally we take a to be an atom, since that produces fewer terms in (25) than choosing any $b > a$. As an example, let $L = B_n$, the boolean algebra of all subsets of $[n]$, and let $a = \{n\}$. There are two elements $x \in B_n$ such that $x \vee a = \hat{1} = [n]$, viz., $x_1 = [n-1]$ and $x_2 = [n]$. Hence $\mu(x_1) + \mu(x_2) = 0$. Since $[\hat{0}, x_1] = B_{n-1}$ and $[\hat{0}, x_2] = B_n$, we easily obtain $\mu_{B_n}(\hat{1}) = (-1)^n$, agreeing with (4).

If $x \leq y$ in a graded lattice L, write $\mathrm{rk}(x, y) = \mathrm{rk}(y) - \mathrm{rk}(x)$, the length of every saturated chain from x to y. The next result may be stated as "the Möbius function of a geometric lattice strictly alternates in sign."

Theorem 3.10. *Let L be a finite geometric lattice with Möbius function μ, and let $x \leq y$ in L. Then*

$$(-1)^{\mathrm{rk}(x,y)} \mu(x, y) > 0.$$

Proof. Since every interval of a geometric lattice is a geometric lattice (Exercise 3), it suffices to prove the theorem for $[x, y] = [\hat{0}, \hat{1}]$. The proof is by induction on the rank of L. It is clear if $\mathrm{rk}(L) = 1$, in which case $\mu(\hat{0}, \hat{1}) = -1$. Assume the result for geometric lattices of rank $< n$, and let $\mathrm{rk}(L) = n$. Let a be an atom of L in Theorem 3.9. For any $y \in L$ we have by semimodularity that

$$\mathrm{rk}(y \wedge a) + \mathrm{rk}(y \vee a) \leq \mathrm{rk}(y) + \mathrm{rk}(a) = \mathrm{rk}(y) + 1.$$

Hence $x \vee a = \hat{1}$ if and only if $x = \hat{1}$ or x is a coatom (i.e., $x \lessdot \hat{1}$) satisfying $a \not\leq x$. From Theorem 3.9 there follows

$$\mu(\hat{0}, \hat{1}) = -\sum_{a \not\leq x \lessdot \hat{1}} \mu(\hat{0}, x).$$

The sum on the right is nonempty since L is atomic, and by induction every x indexing the sum satisfies $(-1)^{n-1}\mu(\hat{0}, x) > 0$. Hence $(-1)^n \mu(\hat{0}, \hat{1}) > 0$. $\square$

Combining Proposition 3.8 and Theorem 3.10 yields the following result.

Corollary 3.4. *Let $\mathcal{A}$ be any arrangement and $x \leq y$ in $L(\mathcal{A})$. Then*

$$(-1)^{\mathrm{rk}(x,y)}\mu(x, y) > 0,$$

where μ denotes the Möbius function of $L(\mathcal{A})$.

Similarly, combining Theorem 3.10 with the definition (22) of $\chi_M(t)$ gives the next corollary.

Corollary 3.5. *Let M be a matroid of rank n. Then the characteristic polynomial $\chi_M(t)$ strictly alternates in sign, i.e., if*

$$\chi_M(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0,$$

then $(-1)^{n-i}a_i > 0$ for $0 \leq i \leq n$.

Let $\mathcal{A}$ be an n-dimensional arrangement of rank r. If $M_{\mathcal{A}}$ is the matroid corresponding to $\mathcal{A}$, as defined in Proposition 3.6, then

$$\chi_{\mathcal{A}}(t) = t^{n-r}\chi_M(t). \tag{26}$$

It follows from Corollary 3.5 and equation (26) that we can write

$$\chi_{\mathcal{A}}(t) = b_n t^n + b_{n-1}t^{n-1} + \cdots + b_{n-r}t^{n-r},$$

where $(-1)^{n-i}b_i > 0$ for $n - r \leq i \leq n$.