Course 18.312: Algebraic Combinatorics

Lecture Notes # 15 Addendum by Gregg Musiker

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The material for this lecture can be found in several sources, for example see Section 4.1 of William Fulton's book "Young Tableaux".

1 Proof of Schensted's Theorem

Theorem (Schensted). Let π be a permutation of $\{1, 2, \ldots, n\}$ written in one-line notation. Let P and Q be the Standard Young Tableaux (SYT) in the image of the (Robinson-Schensted-Knuth) RSK algorithm, i.e. $RSK(\pi) = (P, Q)$, with shapes $sh(P) = sh(Q) = \lambda$. Then the length of the longest **increasing subsequence** in π equals the length of the **first row of** λ and the length of the longest **decreasing subsequence** in π equals the length of the **first column** of λ.

Before proving this theorem, we start with a few Lemmas.

Lemma 1: Let P_{k-1} be the NYT constructed by inserting $\pi(1)$, $\pi(2)$,..., $\pi(k-1)$. (Note that we do not have a fully Standard Young Tableau because we have only done partial insertion. Instead we have a "Near" Young Tableau.) Assume that the entry $\pi(k)$ about to be inserted will go into the jth column of tableau P_k. Then, the longest increasing subsequence of π ending with $\pi(k)$ has length j.

Proof. (By induction on k) The claim is clear for the case $k = 1$. Assume row 1 of P_{k-1} has a y in column $(j-1)$ and a z in column j. Then if $\pi(k)$ is inserted into column j, this implies the inequalities $y < \pi(k) < z$. Additionally, $y = \pi(r)$ for some $r \leq k-1$ and when $\pi(r)$ is inserted, it was inserted into column $(j-1)$. This implies that there is an increasing subsequence of length $(j-1)$ in the set $\{\pi(1), \ldots, \pi(r)\}$ ending with $\pi(r)$. Furthermore, $\pi(r) < \pi(k)$, so there exists an increasing subsequence of length j ending with $\pi(k)$.

Now we need to show that there is no longer subsequence ending with $\pi(k)$. If there were, that would involve

$$\pi(i_1) < \cdots < \pi(i_p) < \pi(k)$$

where $i_p < k$. Assume that the index p is chosen so that there are no elements between $\pi(i_p)$ and $\pi(k)$. Thus $\pi(1), \ldots, \pi(i_p)$ would contain a subsequence of length greater than or equal to j. By induction, this would also mean that P_{i_p} would be built by **inserting** element $\pi(i_p)$ to the right of the $(j-1)$st column in the first row. Since no $\pi(i)$, for $i_p < i < k$ satisfies $\pi(i) > \pi(i_p)$, it follows that P_{k-1} contains $\pi(i_p)$ in the jth column or further to the right. However, then $\pi(i_p) = z$ or to its right, and either way we would obtain $\pi(k) < \pi(i_p)$ and a contradiction.

Corollary. The longest increasing subsequence in π is the length of the first row in $P(\pi)$, which equals λ_1.

Let π^{rev} denote the reverse of π. For example, if $\pi = 5\ 1\ 4\ 2\ 3$, then $\pi^{rev} = 3\ 2\ 4\ 1\ 5$.

Proposition (Schensted). If $P(\pi) = P$ then $P(\pi^{rev}) = P^T$, the conjugate SYT.

Sketch of Proof. The proof works by using **column insertion** instead of row insertion. This is sometimes referred to as **Dual RSK**. In particular, we *insert* blocks into the first column as *low* as possible and then *bump* elements to columns to the right. If one applies row insertion and column insertion to the same permutation π, and then compares the outputs, then one sees that step-by-step one gets tableaux that are transposes of one another. The **Key Lemma** is that for any P_k and $x, y \notin P_k$ (i.e. x and y come later in the permutation π), then

$$c_y r_x(P_k) = r_x c_y(P_k).$$

Here c_y denotes the column insertion of element y and r_x denotes the row insertion of element x. The proof of the Key Lemma can be found in "Young Tableaux" and is omitted.

Corollary 1. If $RSK(\pi) = (P, Q)$, then $RSK(\pi^{rev}) = (P^T, Q^T)$.

Proof. If $RSK(\pi) = (P, Q)$ then $P = r_{\pi_k} \circ r_{\pi_{k-1}} \circ \cdots \circ r_{\pi_2} \circ r_{\pi_1}(\emptyset)$, acting by row insertion on the left. Consequently,

$$\begin{aligned} P^{rev} &= r_{\pi_1} \circ r_{\pi_2} \circ \cdots \circ r_{\pi_{k-1}} \circ r_{\pi_k}(\emptyset) \\ &= r_{\pi_1} \circ r_{\pi_2} \circ \cdots \circ r_{\pi_{k-1}} \circ c_{\pi_k}(\emptyset). \end{aligned}$$

We rearrange these last two entries by the Key Lemma to obtain

$$\begin{aligned} P^{rev} &= r_{\pi_1} \circ r_{\pi_2} \circ \cdots \circ r_{\pi_{k-2}} \circ c_{\pi_k} \circ r_{\pi_{k-1}}(\emptyset) \\ &= c_{\pi_k} \circ r_{\pi_1} \circ r_{\pi_2} \circ \cdots \circ r_{\pi_{k-2}} \circ r_{\pi_{k-1}}(\emptyset), \end{aligned}$$

and we can also change $r_{\pi_{k-1}}$ into a $c_{\pi_{k-1}}$ by the same logic as above. Proceeding iteratively and moving the operator c_{π_i} leftwards, we obtain

$$\begin{aligned} P^{rev} &= c_{\pi_k} \circ c_{\pi_{k-1}} \circ \cdots \circ c_{\pi_2} \circ c_{\pi_1}(\emptyset) \\ &= P^T. \end{aligned}$$

Since this result is true for partial tableaux, we obtain that $Q^{rev} = Q^T$ for the **recording tableaux** as well.

Corollary 2. The length of the longest **decreasing** subsequence equals the length of the longest **column**.

This completes the proof of Schensted's Theorem, as well as the fact that $RSK(\pi^{rev}) =$ Dual RSK(π).

Symmetry Theorem (Schutzenberger). If $RSK(\pi) = (P, Q)$ then $RSK(\pi^{-1}) = (Q, P)$.

Proof Omitted.

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