Course 18.312: Algebraic Combinatorics

Problem Set # 3

Due Wednesday February 25th, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) (5 points) Find the unique four-element poset which is not a series-parallel poset.

   (10 points) Prove that the map $\mu : (B_n)_i \to (B_n)_{i+1}$ defined on page 26 of Section 4 of the notes (and in class on Wednesday) is injective when $i < n/2$, and thus an order matching.

2) (5 points) Show that for $(x, y) \leq (x', y') \in P \times Q$ that

$$\mu_{P \times Q}\Big((x, y), (x', y')\Big) = \mu_P(x, x')\mu_Q(y, y').$$

   For the purposes of this problem, let C_k denote a chain of length k.

   (10 points) Show that the Möbius function on the boolean poset B_n is given by the formula

$$\mu_{B_n}(S, T) = (-1)^{|T|-|S|}$$

   whenever $S \subset T \subset \{1, 2, \ldots, n\}$.

   (**Hint:**) Show that the boolean poset B_n is isomorphic to the poset $(C_2)^n = C_2 \times \ldots \times C_2$.

   (5 points) Show that if positive integer n factors as $n = p_1^{e_1} \cdots p_m^{e_m}$, (the p_i's are prime) then the poset of divisors of n, D_n is isomorphic to $C_{e_1+1} \times \cdots \times C_{e_m+1}$.

   (10 points) Show that $\mu_{D_n}(1, d)$ agrees with the the number theoretic Möbius function $\hat{\mu}(d)$ defined in Lecture Notes from February 18th.

3) Let $G = \{1, \pi\}$ be a group of order two (with identity element 1). Let G act on $\{1, 2, 3, 4\}$ by $\pi \cdot 1 = 1$, $\pi \cdot 2 = 3$, $\pi \cdot 3 = 2$, $\pi \cdot 4 = 4$.

(5 points) Draw the Hasse diagram of the quotient poset B_4/G.

(5 points) What is the size of the largest antichain? List all antichains of this size.

(10 points) Draw the Hasse diagram and do the same computations for the action $\pi \cdot 1 = 4$, $\pi \cdot 2 = 3$, $\pi \cdot 3 = 2$, $\pi \cdot 4 = 1$.

4) A $(0, 1)$-necklace of length n and weight i is a circular arrangement of i 1's and $(n - i)$ 0's. For instance the $(0, 1)$-necklaces of lengths 6 and weight 3 are (writing a circular arrangement linearly) 000111, 001011, 010011, and 010101. (Cylcic shifts of a linear word represent the same necklace, e.g., 000111 is the same as 110001.) Let N_n denote the set of all $(0, 1)$-necklaces of length n. Define a partial order on N_n by letting $u \leq v$ if we can obtain v from u by changing some 0's to 1's. It is easy to see (you may assume it) that N_n is graded of rank n, with the rank of a necklace being its weight.

(15 pts) Show that N_n is rank-symmetric, rank-unimodal, and Sperner.

Hint: Show that $N_n \cong B_n/G$ for a suitable group G.

5) Define the **shift** of a linear word $a_1 a_2 \ldots a_n$ to be the linear word $a_n a_1 a_2 \ldots a_{n-1}$, and define the **period** to be the smallest number of shifts needed to return to the original word. For example, the period of 010101 is 2, the period of 101101 is 3, while the period of 001011 is 6.

$$(\#\text{ number of necklaces of length } n) = \sum_{d|n} c_d \ (\#\text{ strings of period } d)$$

for some choice of c_d's.

(5 points) Show this identity with the proper c_d's filled in.

(5 points) What is a closed expression for $\sum_{d|n} (\#\text{ strings of period } d)$?

(10 points) Use Möbius inversion to obtain a closed formula for the number of necklaces of length n.