18.306 Advanced Partial Differential Equations with Applications
Fall 2009

Problem Set Number 05

18.306 — MIT (Fall 2009)

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Contents

1.1 Statement: Wave equations (problem 01).

Consider an elastic (homogeneous) string under tension, tied at one end, initially at rest, and forced by a (small amplitude) harmonic shaking of the other end. To simplify the situation, assume that all the motion is restricted to happen in a plane.

After a proper adimensionalization, the situation is modeled by the mathematical problem below for the wave equation in 1-D — where $u = u(x, t)$ is the displacement from equilibrium of the string.

$$u_{tt} - u_{xx} = 0, \quad \text{for} \quad 0 < x < 1, \quad \text{and} \quad t > 0, \tag{1.1}$$

with *initial data* $u(x, 0) = u_t(x, 0) = 0$, and *boundary conditions*

$$u(0, t) = 1 - \cos(\omega\, t) \quad \text{and} \quad u(1, t) = 0. \tag{1.2}$$

FIND the solution to this problem, for the times $0 < t \leq 4$. Furthermore: note that the solution, while making sense in the classical sense (no need to invoke generalized function derivatives), is not infinitely differentiable. There are certain lines along which "singularities" occur. **FIND** these lines of singularity, and describe what the situation is along them (nature of the singularities) — the lines are, of course, characteristics. **FINALLY:** does anything special happen if $\omega = \pi$?

1.2 Statement: Wave equations (problem 02).

Consider an elastic (homogeneous) string under tension, undergoing small amplitude oscillations, and assume that all the motion is restricted to happen in a plane. Under these conditions, and after a proper adimensionalization, the displacements $u = u(x, t)$ from equilibrium of the string can be shown to satisfy the 1-D wave equation

$$u_{tt} - u_{xx} = 0, \tag{1.3}$$

to which appropriate *initial data* and *boundary conditions* must be applied to determine a unique solution.

In the lectures we showed that the second order (in space and time) equation in (1.3) is equivalent to a system of two first order equations. We did this by introducing the variables $v = u_t$ and $w = u_x$, for which it can be seen that

$$v_t - w_x = 0 \quad \text{and} \quad w_t - v_x = 0 \tag{1.4}$$

apply — the first equation is (1.3) and the second follows from equality of cross-derivatives. On the other hand, the second equation in (1.4) guarantees that there is a u such that $v = u_t$ and $w = u_x$, and then the first equation yields (1.3).

Consider now the situation where the string is attached to an (homogeneous) "elastic bed", instead of being free in space.[1] In this case, in addition to the forces caused by the tension in the string, forces are generated by the bed — which are functions of the displacements u only. Thus the

[1] For example: imagine a ribbon made of some elastic material, with one edge attached to a rigid surface, the other edge attached to the string, and thin enough that we can ignore its mass.

governing equation above in (1.3) must be modified to

$$u_{tt} - u_{xx} + g(u) = 0, \tag{1.5}$$

where g characterizes the elastic response by the bed. If Hooke's law applies, then $g = \kappa\, u$ — for some elastic constant $\kappa > 0$.

By introducing appropriate variables, **SHOW THAT** *the second order equation in (1.5) is equivalent to a first order system of two equations in two unknowns functions.*

Hint: because the function u appears in equation (1.5), the trick that we used for (1.3) does not work for (1.5). If you introduce $v = u_t$ and $w = u_x$ as new variables, you will also have to keep u, and then you will end up with three variables (not two). Instead, try introducing as a new variable an appropriate combination of u_t and u_x.

Note that the approach that you develop here should work for any $g = g(u)$. In particular, for $g \equiv 0$ it will give you a *different* (from the one used in the lectures) way to show that (1.3) is equivalent to a system of two first order equations.

1.3 Statement: Homogenization (problem 01). Slow compression.

Consider the slow compression/decompression by a piston of a gas in a closed container. On the one hand, intuition tells us that gas properties (such the density) should be uniform across the container, varying in time (only) to adjust for the volume changes. On the other hand, the behavior should be governed by the equations of gas dynamics, which tells us that changes in the gas state happen only through waves propagating at the speed of sound. Since these waves should start at the piston, the solution cannot be just a function of time. Here we show, with a simple model problem, that *these two, seemingly contradictory scenarios, coexist without trouble: they are BOTH correct!*

Consider a gas in a closed cylindrical pipe, where one of the ends is closed by a piston, that can be moved to change the pipe length. Let us neglect motion of the gas in the directions perpendicular to the pipe axis, neglect viscous forces and thermal conductivity, and approximate the evolution as being at constant entropy (adiabatic).[2] Thus we arrive at the **equations**

$$\rho_t + (\rho\, u)_x = 0 \quad \text{and} \quad u_t + u\, u_x + \frac{1}{\rho}\, p_x = 0, \quad \text{for } \; 0 < x < a \; \text{ and } \; t > 0, \tag{1.6}$$

where ρ is the density, u is the flow velocity, $p = p(\rho)$ is the pressure, $c = c(\rho) = \sqrt{dp/d\rho} > 0$ is the sound speed, and a is the varying length of the pipe (piston position). We assume non-dimensional variables, related to the dimensional variables (denoted with tildes) via

$$\tilde{x} = L\, x, \quad \tilde{t} = \frac{L}{c_0}\, t, \quad \tilde{a} = L\, a, \quad \tilde{\rho} = \rho_0\, \rho, \quad \tilde{u} = c_0\, u, \quad \tilde{c} = c_0\, c, \quad \text{and} \quad \tilde{p} = \rho_0\, c_0^2\, p, \tag{1.7}$$

where L is the "average" pipe length, ρ_0 is a typical density, and c_0 is the sound speed corresponding to ρ_0 — note, in particular, that: $\boldsymbol{L/c_0}$ is the time a "typical" sound wave takes to go from one end of the pipe, to the other end. Let now t_p be the time over which the piston position changes significantly. Hence, the statement that the piston moves "slowly" can be phrased as follows

$$a = a\left(\frac{\tilde{t}}{t_p}\right) = a(\epsilon\, t), \quad \text{where } \; 0 < \epsilon = \frac{L}{c_0\, t_p} \ll 1, \tag{1.8}$$

and **the function $\boldsymbol{a = a(\tau)}$ is order one size with order one size derivatives.** Note that $\epsilon \ll 1$ is equivalent to:

Sound waves bounce back and forth in the pipe a large number of times,
in the period it takes the pipe length to vary by a significant fraction. (1.9)

The **boundary conditions for (1.6) are then:**

$$u = 0 \quad \text{at } \; x = 0 \qquad \text{and} \qquad u = \epsilon\, b(\epsilon\, t) \quad \text{at } \; x = a(\epsilon\, t), \tag{1.10}$$

where $b = b(\tau) = \dfrac{da}{d\tau}$.

Part 1. Simple Linearization.

Consider the initial value problem for (1.6) and (1.10), where the initial density is uniform, and the gas is at rest. Hence, without loss of generality (because of the way we selected the nondimensional variables), we can write

$$\rho(x, 0) = 1, \quad \text{and} \quad u(x,\, 0) = 0, \tag{1.11}$$

[2] These approximations are coarse, but they serve the purpose of giving a simple model that we can analyze.

as well as $\boldsymbol{a(0) = c(1) = 1}$ and $\boldsymbol{b(0) = \mu = \pm 1}$ — $\mu = -1$ is compression and $\mu = 1$ is expansion. Linearize the problem, by: (i) Expand $\rho = 1 + \epsilon\,\rho_1 + O(\epsilon^2)$, $u = \epsilon\,u_1 + O(\epsilon^2)$, $a = 1 + O(\epsilon)$, and $b = \mu + O(\epsilon)$. (ii) Substitute into the equations, and neglect higher order contributions. This gives a linear acoustics, initial-boundary value problem in $\boldsymbol{0 < x < 1}$ and $\boldsymbol{t > 0}$, whose solution can be written explicitly. **Tasks:**

- **1a. Write the linear acoustic problem stated above** (derive the equations, the initial conditions and the boundary conditions).
- **1b. Solve the problem** (use characteristics).
- **1c. Describe what happens with the solution for large times. Show that the linearization above breaks down in this limit.**
- **1d. For which range of times is the linearization above valid? Show that a restriction of the form $0 \leq \boldsymbol{t} \ll \boldsymbol{\epsilon^{-p}}$ applies, for some $\boldsymbol{p} > 0$. Find $\boldsymbol{p}$.**

Hints. (a) *You need to expand the boundary condition at the piston location, namely:* $u(a,\,t) = \epsilon\,b$. **(b)** *Find the general solution to the equations, and express it in terms of two arbitrary functions* $f = f(t-x)$ *and* $g = g(t+x)$. **(c)** *Substitute the general solution into the initial and boundary conditions, and determine* f *and* g. **(d)** *The initial conditions determine* $f = f(\xi)$ *and* $g = g(\xi)$ *for some range of* ξ. *Then the boundary conditions will allow you to determine* f *and* g *for larger and larger ranges of* ξ, *each step corresponding to one travel time — back and forth — of a sound wave across the pipe.* **(e)** *You should be able to use the boundary conditions to write* g *in terms of* f, *so that the full solution depends on a single function* f, *which you should be able to write explicitly.* **(e)** *In fact, it is possible to write a recursion relation, that gives* $f(\xi+2)$ *in terms of* $f(\xi)$. *Then write* f *as a sum of a linear function and a periodic function, and thus find* f *for all values of* ξ. **(f)** *Look at the solution you just obtained, and notice that the linearization under which it was derived assumes that* $\epsilon\,\rho_1$ *is small. Is there a problem?*

Part 2. Homogenization. Here we show how to produce an approximation that does not break down for large times. **This part will be assigned with the next problem set.**

THE END.