18.306 Advanced Partial Differential Equations with Applications
Fall 2009

18.306 Problem List.

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Abstract

Problem list for 18.306. These problems may be assigned in problem sets and/or exams.

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1 Linear First Order PDE.

1.1 Statement: Linear 1st order PDE (problem 01).

Part 1. Find the general solutions to the two 1st order linear scalar PDE

$$x\,u_x + y\,u_y = 0, \qquad \text{and} \qquad y\,v_x - x\,v_y = 0. \tag{1.1}$$

Hint: *The general solutions take a particular simple form in polar coordinates.*

Part 2. For u, find the solution such that on the circle $x^2 + y^2 = 2$, it satisfies $u = x$. *Where is this solution determined by the data given?*

Part 3. Is there a solution to the equation for v such that $v(x, 0) = x$, for $-\infty < x < \infty$?

Part 4. How does the general solution for u changes if the equation is modified to

$$x\,u_x + y\,u_y = (x^2 + y^2)\,\sin(x^2 + y^2)? \tag{1.2}$$

1.2 Statement: Linear 1st order PDE (problem 02).

Consider the following problem

$$x\,u_x + y\,u_y = 1 + y^2, \qquad \text{with} \qquad u(x,\,1) = 1 + x \quad \text{for} \quad -\infty < x < \infty. \tag{1.3}$$

Part 1. Use the method of characteristics to solve this problem. Write the solution $u = u(x,\,y)$ (explicitly) as a function of x and y on $y > 0$.

Part 2. Explain why $u = u(x,\,y)$ is not uniquely determined by the problem above for $y \leq 0$ — you may use a diagram.

1.3 Statement: Linear 1st order PDE (problem 03).

Consider the following problem

$$u_x + 2\,x\,u_y = y, \qquad \text{with} \qquad u(0,\,y) = f(y) \quad \text{for} \quad -\infty < y < \infty, \tag{1.4}$$

where $f = f(y)$ is an "arbitrary" function.

Part 1. Use the method of characteristics to solve this problem. Write the solution $u = u(x,\,y)$ as a function of x, y, and f. In which part of the $(x,\,y)$ plane is the solution uniquely determined?

Part 2. Let f have a continuous derivative. Are then the partial derivatives u_x and u_x continuous?

1.4 Statement: Linear 1st order PDE (problem 04).

Discuss the two problems

$$u_x + 2\,x\,u_y = y, \qquad \text{with} \qquad \begin{cases} \text{(a) } u(x,\,x^2) = 1 & \text{for } \ -1 < x < 1, \\ \text{(b) } u(x,\,x^2) = \frac{1}{3}\,x^3 + \pi & \text{for } \ -1 < x < 1. \end{cases} \tag{1.5}$$

How many solutions exist in each case?

Note that the data in these problems is prescribed along a characteristic!

1.5 Statement: Linear 1st order PDE (problem 05).

Consider the problem

$$x\,u_x + (x+y)\,u_y = 1, \qquad \text{with} \qquad u(1,\,y) = y \ \text{ for } \ 0 < y < 1. \tag{1.6}$$

Question 1. Write the solution $u = u(x,\,y)$ in the region where it is uniquely determined.

Question 2. Describe the region in the plane where the solution to (1.6) is uniquely determined.

Question 3. Write all the functions $u = u(x,\,y)$ that satisfy (1.6) on $x > 0$ and $-\infty < y < \infty$.

Question 4. Write all the functions $u = u(x,\,y)$ that satisfy the pde in (1.6) for $x < 0$.

Question 5. What happens along $x = 0$? Can you produce solutions to the pde that are continuous in the "punctured" plane (plane minus the origin)?

1.6 Statement: Linear 1st order PDE (problem 06).

Consider the pde

$$u_x + 2\,x\,u_y = y + x\,u. \tag{1.7}$$

Part 1. Write the characteristic form for this equation, and use it to write the general solution $u = u(x,\,y)$ to the pde — the general solution should involve an arbitrary function $f = f(\cdot)$.

Part 2. Find $u = u(x,\,y)$ if $u = u(0,\,y) = 1 + y^2$ for $1 < y < 2$. In which region is the solution u uniquely defined by this?

1.7 Statement: Linear 1st order PDE (problem 07).

A function $u = u(x,\,y)$ is called *homogeneous of degree* $n > 0$ if and only if $u(\lambda\,x,\,\lambda\,y) = \lambda^n\,u(x,\,y)$, for any constant λ, over the range of independent variables for which u is defined.

Part 1. Consider the homogeneous functions of degree n, defined on the right hand plane $x > 0$, and obtain a pde that all such functions must satisfy. [**Hint:** differentiate the equation satisfied by u, and set $\lambda = 1$.]

Part 2. Use the method of characteristics to find the general solution of the pde derived in part 1, and show that all its solutions are homogeneous functions of degree n.

1.8 Statement: Discontinuous Coefficients in Linear 1st order pde #01.

Singularities (in particular, discontinuities) in the coefficients of a pde can create ambiguities in the meaning[1] of the equation. Sometimes these ambiguities can be easily resolved, and other times they cannot. In all cases, however, it is advisable to go back to the physical system that the pde is supposed to model, and either (a) Check that the meaning given to the solutions across the singularities in the coefficients makes physical sense, or (b) Seek for the meaning, if not clear, there.

In this exercise we consider a simple example of the situation described in the prior paragraph. Consider the initial value problem

$$u_t + c\,u_x = \mathrm{sign}(x) \quad \text{for } \ t > 0 \ \text{ and } \ -\infty < x < \infty, \quad \text{with } \ u(x,\,0) = g(x), \tag{1.8}$$

where c is some constant and g is some arbitrary function — which we will assume, for simplicity, is smooth. **The task is now to give a unique, unambiguous, meaning to this I.V.P.**

An important (practical) consideration is that the solution to any mathematical question must not be sensitive to small changes in the formulation of the problem. This implies that

$$\left.\begin{array}{l}\textit{The solution to the problem in (1.8) should not change very much } \text{[for any finite time} \\ \text{interval } 0 < t < T\,] \textit{ if either } c\textit{, or the forcing function } \mathrm{sign}(x)\textit{, are modified slightly.}\end{array}\right\} \tag{1.9}$$

We will use this requirement to give a clear meaning to the problem in (1.8), as follows.

Step 1. Replace (1.8) by the set of problems, parameterized by $\epsilon > 0$,

$$u_t + c\,u_x = f_\epsilon(x) \quad \text{for } \ t > 0 \ \text{ and } \ -\infty < x < \infty, \quad \text{with } \ u(x,\,0) = g(x), \tag{1.10}$$

where f_ϵ is a smooth, non-decreasing, function satisfying $f_\epsilon(x) = \mathrm{sign}(x)$ for $|x| > \epsilon$. **Show that the limit $\epsilon \downarrow 0$ of the solutions to (1.10) exists, and it is independent of the choice of the functions f_ϵ.**

[1] What does it mean to be a solution?

Thus we can use the $\epsilon \downarrow 0$ limit of (1.10) to give a clear meaning to (1.8).

Hint. Write the solution to (1.10) using characteristics, and then take the limit $\epsilon \downarrow 0$.

Step 2. Show that the solution obtained in step 1 is not sensitive to small changes in the initial data $g = g(x)$, or to changes in the value of c — as long as $c \neq 0$. **What happens for** $c \approx 0$?

1.9 Statement: Discontinuous Coefficients in Linear 1st order pde #02.

Singularities (in particular, discontinuities) in the coefficients of a pde can create ambiguities in the meaning[2] of the equation. Sometimes these ambiguities can be easily resolved, and other times they cannot. In all cases, however, it is advisable to go back to the physical system that the pde is supposed to model, and either (a) Check that the meaning given to the solutions across the singularities in the coefficients makes physical sense, or (b) Seek for the meaning, if not clear, there.

In this exercise we consider an example of the situation described in the prior paragraph. **The task is to give a unique, unambiguous, meaning to the following initial value problem (I.V.P.)**

$$u_t + \mathrm{sign}(x)\, u_x = g(x) \quad \text{for } \; t > 0 \; \text{ and } \; -\infty < x < \infty, \quad \text{with } \; u(x,\,0) = U(x), \tag{1.11}$$

where g and U are "arbitrary" smooth functions. An important (practical) consideration is that well posed questions do not have answers sensitive to small changes in problem formulation. Hence

$$\left.\begin{array}{l} \textit{The solution to the problem in (1.11) should not change very much } [\text{for any finite time} \\ \text{interval } 0 < t < T\,] \textit{ if either } g\textit{, or the coefficient function } \mathrm{sign}(x)\textit{, are modified slightly.} \end{array}\right\} \tag{1.12}$$

We use this requirement to give a clear meaning to the problem in (1.11), as follows.

STEP 1. Replace (1.11) by the set of problems, parameterized by $\epsilon > 0$,

$$u_t + f_\epsilon(x)\, u_x = g(x) \quad \text{for } \; t > 0 \; \text{ and } \; -\infty < x < \infty, \quad \text{with } \; u(x,\,0) = U(x), \tag{1.13}$$

where f_ϵ is a smooth, non-decreasing, function satisfying $f_\epsilon(x) = \mathrm{sign}(x)$ for $|x| > \epsilon$. **Show that** the limit $\epsilon \downarrow 0$ of the solutions to (1.13) exists, and it is independent of the choice of the functions f_ϵ.

Hint: write the problem (1.13) in characteristic form, and consider what happens with the characteristics as $\epsilon \downarrow 0$. Drawing them in the space time diagram should be helpful.

[2] What does it mean to be a solution?

Thus we can use the $\epsilon \downarrow 0$ limit of (1.13) to give a clear meaning to (1.11).

Hint/warning: be careful with the limit in the region $|x| \leq t$! The best formulation of the limit is as **two signaling/initial value problems:** one for x, $t > 0$ and another one for $x < 0 < t$ — each with appropriate boundary conditions on $x = 0$, $t > 0$.

STEP 2. Show that the solution obtained in step 1 is not sensitive to small changes in the functions $g = g(x)$ or $U = U(x)$.

1.10 Statement: Discontinuous Coefficients in Linear 1st order pde #03.

In the prior two problems (Discontinuous Coefficients in Linear 1st order pde #01 and #02) we considered a first order, scalar, linear pde in 1-D space-time with discontinuous coefficients. In problem #01 the discontinuity was introduced in the source term, while in problem #02 the discontinuity was introduced in the characteristic speed. In both cases we showed that the solution can be defined without ambiguities as the limit of the solutions for the problems where the discontinuities are eliminated by "smearing" the discontinuous coefficient over a small interval.

In this problem we will consider the situation where **discontinuities** are introduced in **both** the characteristic speed and in the source term. Namely, consider the following initial value problem

$$u_t + \mathrm{sign}(x)\, u_x = \mathrm{sign}(x) \quad \text{for } \ t > 0 \ \text{ and } \ -\infty < x < \infty, \quad \text{with } \ u(x,\, 0) = U(x), \tag{1.14}$$

where U is an "arbitrary" smooth function. We now ask the question: **Is it possible to give a unique, unambiguous,meaning to this I.V.P., following the same approach used in the two prior exercises?** That is, consider the set of problems (parameterized by $\epsilon > 0$) given by

$$u_t + f_\epsilon(x)\, u_x = g_\epsilon(x) \quad \text{for } \ t > 0 \ \text{ and } \ -\infty < x < \infty, \quad \text{with } \ u(x,\, 0) = U(x), \tag{1.15}$$

where f_ϵ and g_ϵ are smooth, non-decreasing, functions satisfying $f_\epsilon(x) = g_\epsilon(x) = \mathrm{sign}(x)$ for $|x| > \epsilon$. Then we would like to show that the limit $\epsilon \downarrow 0$ of the solutions to (1.15) exists, and that it is independent of the choice of the functions f_ϵ and g_ϵ. **Show that this is false.** Namely: the limit $\epsilon \downarrow 0$ of the solutions to (1.15), if any, depends on the particular selection of the functions f_ϵ and g_ϵ. In particular, the answer to the question above is: **No, it is not possible to give a unique, unambiguous, meaning to the I.V.P. in (1.14) without further information about the solutions.**

Remark 1.1 *This does not mean that the problem in (1.14) lacks meaning under all possible circumstances. If extra information about the "physics" behind the problem is known, then a meaning might be attached.* For example: assume that *the discontinuities in the coefficients of (1.14) arise because of approximations to scales that we cannot resolve, but that the scales involved in the source terms are much larger than the scales involved in the wave speeds. This means that the problem can be considered as an approximation to something of the form*

$$u_t + f_\epsilon(x)\, u_x = g_\delta(x) \quad \textit{for } t > 0 \textit{ and } -\infty < x < \infty, \quad \textit{with } u(x,\,0) = U(x), \tag{1.16}$$

where: **(i)** *f_ϵ and g_δ are smooth, non-decreasing, functions satisfying $f_\epsilon(x) = sign(x)$ for $|x| > \epsilon$ and $g_\delta(x) = sign(x)$ for $|x| > \delta$.* **(ii)** $0 < \epsilon \ll \delta \ll 1$. Furthermore, assume that *the limit of $g_\delta(0)$ as $\delta \downarrow 0$ exists.* **In this case the problem in (1.14) can be given a meaning as follows:** *First, take the limit $\epsilon \downarrow 0$ of (1.16) — this as it is done in the problem "Discontinuous Coefficients in Linear 1st order pde #02". Then take the limit $\delta \downarrow 0$ — this is trivial.*

Hint: in order to do this exercise, take a look at the answer to the problem *Discontinuous Coefficients in Linear 1st order pde #02*, and find where the arguments there go wrong. This should give you a pretty good idea of what is wrong with the limit $\epsilon \downarrow 0$ in (1.15).

PART 2. Let the (continuous) function $S = S(x)$ be defined by

$$\mathcal{S}(x) = \text{sign}(x) \quad \text{for } |x| \geq 1, \quad \text{and} \quad \mathcal{S}(x) = x \quad \text{for } |x| \leq 1. \tag{1.17}$$

Calculate the limit $\epsilon \downarrow 0$ of (1.15) in the following two cases — you should get different answers!
Case 1: $f_\epsilon(x) = \mathcal{S}(x/\epsilon)$ and $g_\epsilon(x) = \mathcal{S}(x/\epsilon^2)$.
Case 2: $f_\epsilon(x) = \mathcal{S}(x/\epsilon^2)$ and $g_\epsilon(x) = \mathcal{S}(x/\epsilon)$.

1.11 Statement: Discontinuous Coefficients in Linear 1st order pde #04.

The results of the prior exercise (Discontinuous Coefficients in Linear 1st order pde #03) would seem to indicate that the following initial value problem

$$u_t + \text{sign}(x)\, u_x = a\, \delta(x)\, u \quad \text{for } t > 0 \text{ and } -\infty < x < \infty, \quad \text{with } u(x,\,0) = U(x), \tag{1.18}$$

— where U is an "arbitrary" smooth function, $\delta(\cdot)$ is Dirac's delta function, and a is a constant — has little or no chance of being meaningful.[3] This is generally true. However, **consider the special case when** $a = -2$. Then, at least formally — since $d\,\mathrm{sign}(x)/dx = 2\,\delta(x)$ — (1.18) is equivalent to

$$u_t + (\mathrm{sign}(x)\,u)_x = 0 \quad \text{for} \;\; t > 0 \;\; \text{and} \;\; -\infty < x < \infty, \quad \text{with} \;\; u(x,\,0) = U(x). \tag{1.19}$$

Assume now that u *is the density of some conserved quantity — in particular,* u *must be non-negative:* $u \geq 0$ *— with flux given by* $Q = sign(x)\,u$. Then **show that (1.19) has a unique, un-ambiguous meaning, and write an explicit formula for the solution.**

Hint: The equation has a clear meaning in each of the regions $x,\,t > 0$ and $x < 0 < t$. Thus pose the problems in each of these regions, in terms of the (unknown) values of the solution along the sides of the time axis: $u(0_\pm,\,t) = V_\pm(t)$. Then use the fact that u is a non-negative conserved density to find $V_\pm(t)$ — you will need to invoke the integral form of the conservation law.

1.12 Statement: Discontinuous Coefficients in Linear 1st order pde #05.

The results of the prior exercise (Discontinuous Coefficients in Linear 1st order pde #04) indicate that **it may be possible to assign a unique, unambiguous meaning** to initial value problems of the form

$$u_t + (a(x)\,u + b(x))_x = 0 \quad \text{for} \;\; t > 0 \;\; \text{and} \;\; -\infty < x < \infty, \quad \text{with} \;\; u(x,\,0) = U(x), \tag{1.20}$$

where

a1. $U \geq 0$ is an "arbitrary" smooth function.

a2. $a = a(x)$ is smooth for $x \neq 0$, and has a simple discontinuity at $x = 0$, where a and its derivatives have limits as $x \downarrow 0$ or $x \uparrow 0$. In particular, let the left and right limits of a at $x = 0$ be $\boxed{a_L = \lim_{x \to 0-} a(x)}$ and $\boxed{a_R = \lim_{x \to 0+} a(x),}$ respectively.

a3. $b = b(x)$ is smooth for $x \neq 0$, and has a simple discontinuity at $x = 0$, where b and its derivatives have limits as $x \downarrow 0$ or $x \uparrow 0$. In particular, let the left and right limits of b at $x = 0$ be $\boxed{b_L = \lim_{x \to 0-} b(x)}$ and $\boxed{b_R = \lim_{x \to 0+} b(x),}$ respectively.

[3] The delta function forcing in (1.18) is, clearly, much more singular than a mere discontinuity — which was the subject of the prior exercise.

a4. u is the density of some conserved quantity — in particular, u must be non-negative: $u \geq 0$ — with flux given by $Q = a(x)\,u + b(x)$.

QUESTION: *under which conditions on a_L, a_R, b_L, and b_R can you assign a unique and unambiguous meaning to the problem in (1.20)?* Justify your answer, and explain what goes wrong when the conditions are violated.

Hint 1. The equation has a clear meaning in each of the regions $x,\, t > 0$ and $x < 0 < t$. Thus pose the problems in each of these regions, in terms of the (unknown) values of the solution along the sides of the time axis: $V_L(t) = u(-0,\, t)$ and $V_R(t) = u(+0,\, t)$. Then use the fact that u is a non-negative conserved density function to find $V_L(t)$ and $V_R(t)$ (you will need to invoke the integral form of the conservation law). There are several cases to consider, depending on the signs of a_L, a_R, and $b_R - b_L$ — some with a unique solution, and others with either too many or no solutions.

Make sure that your solution satisfies **CAUSALITY.** In other words: *information must flow along the characteristics FORWARD in time.*

Remark 1.2 *It is important that the equation be compatible with the restriction in* **a4**. *You will need to use this condition when investigating what happens with the solution across $x = 0$. However, it is also important that* **a4** *be compatible with the equation in the regions where a and b are smooth. In other words:* **the characteristic equations should be such that, if u starts non-negative along an arbitrary characteristic, it stays nonnegative for all future times.**

$$\textit{A sufficient condition that guarantees this is } \frac{d\,b}{d\,x} \leq 0. \tag{1.21}$$

Challenge/optional question: *prove this.*

Hint 2. *(i) Along a characteristic, for all times: either $a > 0$, or $a < 0$, or $a = 0$. WHY? (ii) When $a \neq 0$, write a simple equation for $\dfrac{d}{d\,t}(a\,u)$ along the characteristic, which will allow you to track the sign of $(a\,u)$, and show that it does not change.*

2 Semi-Linear First Order PDE.

2.1 Statement: Semi-Linear 1st order PDE (problem 01).

Consider the following problem (here $f = f(y)$ is an "arbitrary" function)

$$u_x + 2\,x\,u_y = 2\,x\,u^2, \qquad \text{with} \qquad u(0,\,y) = f(y) \quad \text{for} \quad -\infty < y < \infty. \tag{2.1}$$

Part 1. Use the method of characteristics to solve this problem. Write the solution $u = u(x,\,y)$ as a function of x, y, and f. Describe the part of the $(x,\,y)$ plane where the solution is determined. What happens if $f \leq 0$ everywhere? What happens when $f > 0$ somewhere?

Part 2. Let $f(y) = y$. Explicitly describe the region where the solution is defined.

2.2 Statement: Semi-Linear 1st order PDE (problem 02).

Discuss the two problems

$$u_x + 2\,x\,u_y = 2\,x\,u^2, \qquad \text{with} \qquad \begin{cases} \text{(a) } u(x,\,x^2) = 1 & \text{for} \quad -1 < x < 1, \\ \text{(b) } u(x,\,x^2) = -\pi/(1+\pi\,x^2) & \text{for} \quad -1 < x < 1, \\ \text{(c) } u(x,\,x^2) = (1-x^2/4)/4 & \text{for} \quad -1 < x < 1. \end{cases} \tag{2.2}$$

How many solutions exist in each case? Where are they uniquely defined?

Note that the data in these problems is prescribed along a characteristic!

3 Kinematic Waves.

3.1 Statement: Conservation Equation in Chromatography.

In chromatography, and similar exchange processes studied in chemical engineering, the following situation arises:

A fluid carrying dissolved substances (or particles, or ions) flows through a fixed bed, and the material being carried is partially absorbed on the fixed solid material in the bed. Let the **fluid flow** be idealized to have a **constant velocity** V. Let ρ_f be the **density of the material carried in the fluid**, and ρ_s be the **density deposited in the solid.** The amount of material being deposited can be related to the amount of material in the fluid by the **exchange equation**, which in its simplest form can be taken to be:

$$(\rho_s)_t = k_1\,(A - \rho_s)\,\rho_f - k_2\,\rho_s\,(B - \rho_f)\,, \tag{3.1}$$

where k_1, k_2, A, and B are constants. The *first term* in this equation represents the deposition from the fluid to the solid, at a rate proportional to the amount in the fluid — but limited by the amount already in the solid, up to a capacity A. The *second term* represents the reverse transfer from the solid to the fluid, at a rate proportional to the amount in the solid — but limited by the amount already in the fluid, up to a capacity B.

At equilibrium $(\rho_s)_t = 0$, and ρ_s is a definite function of ρ_f. In slowly varying conditions, which will arise when the **reaction rates k_1 and k_2 are "large"**, we may still take the approximation in which $(\rho_s)_t = 0$ as far as equation (3.1) is concerned — i.e.: **quasi-equilibrium.**

Remark 3.1 *As usual, the idea here is that, compared with the other time scales in the problem, the reactions leading to equation (3.1) are very fast. Thus, any deviations from the equality relating ρ_s and ρ_f*

$$k_1(A - \rho_s)\rho_f - k_2\rho_s(B - \rho_f) = 0\,, \tag{3.2}$$

that occurs when $(\rho_s)_t = 0$ in (3.1), are very rapidly damped out (so that, at any time, we can assume that the equality above is satisfied.)

Your task: Assuming the quasi-equilibrium approximation, and using the conservation of the material being exchanged, derive an equation for $\rho_s = \rho_s(x,\, t)$.

Hint 3.1 *The total density of the material is $\rho = \rho_s + \rho_f$, while the flow rate follows from knowledge of ρ_f and the (constant) fluid velocity V.*

3.2 Statement: Dispersive Waves and Modulations.

Consider the following linear partial differential equations for the scalar function $u = u(x,\, t)$:

$$\begin{aligned} u_t + c\,u_x + d\,u_{xxx} &= 0, & (3.3)\\ u_{tt} - u_{xx} + a\,u &= 0, & (3.4)\\ i\,u_t + b\,u + g\,u_{xx} &= 0, & (3.5) \end{aligned}$$

where c, d, a, b, and g are constants, and we will assume that the equations have been nondimensionalized.[4] It should be clear that, in all three cases,

$$u = Ae^{i\,(k\,x - \omega\,t)}, \quad \text{where} \quad \omega = \Omega(k)\,, \tag{3.6}$$

is a solution of the equations, for any constants A and k, provided that we take

M1. For equation (3.3): $\Omega(k) = c\,k - d\,k^3$ .... **Verify that this is true.**

M2. For equation (3.4): $\Omega(k) = \sqrt{a + k^2}$ .... **Verify that this is true.**

M3. For equation (3.5): $\Omega(k) = -b + g\,k^2$ .... **Verify that this is true.**

Solutions such as that in (3.6) represent monochromatic traveling waves, with amplitude $|A|$, wave number k, and angular frequency ω. **Note that in all these cases, Ω is NOT a linear function of k.** Thus we say that the **equations are dispersive** and call Ω **the dispersion function.**

Remark 3.2 *For a dispersive system waves with different wavelengths propagate at different speeds. Thus a localized wave packet, made up of many waves of different wavelengths, will* **disperse** *in time — as the waves cease to add up in the proper phases to guarantee a localized wave-packet.*

Your task: consider a dispersive waves system, that is: a system of equations accepting monochromatic traveling waves as solutions, provided that their wave number k and angular frequency ω are related by a dispersion relation

$$\omega = \Omega(k)\,. \tag{3.7}$$

Consider now a slowly varying, nearly monochromatic solution of the system. To be more precise: consider a solution such that at each point in space–time one can associate a *local wave number* $\boxed{k = k(x,\,t)}$ and a *local angular frequency* $\boxed{\omega = \omega(x,\,t).}$ In particular, both k and ω vary slowly in space and time, so that they change very little over a few wavelengths or a few wave periods. *Note though that they may change considerably over many wavelengths or wave periods!* Then

Assuming conservation of wave crests,
derive equations governing k and ω.

These equations are called the **Wave Modulation Equations.**

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[4] These equations arise in many applications, but we will not be concerned with these applications here.

Remark 3.3 *Notice that the assumption that k and ω vary slowly is fundamental in making sense of the notion of a locally monochromatic wave. To even define a wave number or an angular frequency, the wave must look approximately monochromatic over several wavelengths and periods.*

Remark 3.4 **Why is it reasonable to assume that the wave crests are conserved?** *The idea behind this is that, for a wave crest to disappear (or for a new wave crest to appear), something pretty drastic has to happen in the wave field. This is not compatible with the assumption of slow variation. It does not mean that it cannot happen, just that it will happen in circumstances where the assumption of slow variation is invalid. There are some pretty interesting open research problems in pattern formation that are related to this point.*

Hint 3.2 *It should be clear that one of the equations is $\omega = \Omega(k)$, since the solution behaves locally like a monochromatic wave. For the second equation, express the density of wave crests (and its flux) in terms of k and ω. Then write the equation for the conservation of wave crests using these quantities.*

3.3 Statement: Channel Flow Rate Function.

It was shown in the lectures that for a river (or a man-made channel) in the plains, under conditions that are not changing too rapidly (quasi-equilibrium), the following equation should apply

$$A_t + q_x = 0, \tag{3.8}$$

where $A = A(x, t)$ is the cross-sectional filled area of the river bed, x measures length along the river, and $q = Q(A)$ is a function giving the flow rate at any point.

That the flow rate q should be a function of A only[5] follows from the assumption of quasi-equilibrium. Then q is determined by a local balance between the friction forces and the force of gravity down the river bed.

Assume now a man-made channel, with uniform triangular cross-section and a uniform (small) downward slope, characterized by an angle θ. Assume also that the frictional forces are proportional to the product of the flow velocity u down the channel, and the wetted perimeter P_w of the channel bed $F_f = C_f\, u\, P_w$. **Derive the form that the flow function Q should have.**

[5] Possibly also x — i.e. $q = Q(x,\, A)$ — to account for non-uniformities along the river.

Hints: (1) $Q = u\,A$, *where* u *is determined by the balance of the frictional forces and gravity.* (2) *The wetted perimeter* P_w *is proportional to some power of* A.

3.4 Statement: Road capacity.

Consider a road with traffic density obeying the Traffic Flow equation

$$\rho_t + q_x = 0, \tag{3.9}$$

for some flow function $q = Q(\rho)$. How would you determine/measure the *road capacity* $q_m = \max(Q)$ from traffic flow observations on the road?

Hint: *Look at the solution for the red light turns green problem.*

3.5 Statement: Initial Values for a Kinematic Wave (problem 01).

Consider the Kinematic Wave equation

$$u_t + q_x = 0, \qquad \text{where} \qquad q = \frac{1}{2}\,u^2, \tag{3.10}$$

and u is the density for some conserved quantity. Using the method of characteristics, the Rankine-Hugoniot jump conditions, and the entropy conditions, find the solutions (for **all** $t > 0$) to the following initial value problems:

1st Initial Value Problem:

$$u(x,\,0) = \begin{cases} 1 & \text{for} \quad -\infty < x \leq -1, \\ -x & \text{for} \quad -1 \leq x \leq 0, \\ 0 & \text{for} \quad 0 \leq x < \infty. \end{cases} \tag{3.11}$$

2nd Initial Value Problem:

$$u(x,\,0) = \begin{cases} 0 & \text{for} \quad -\infty < x \leq -1, \\ 1+x & \text{for} \quad -1 \leq x \leq 0, \\ 1-x & \text{for} \quad 0 \leq x \leq 1, \\ 0 & \text{for} \quad 1 \leq x < \infty. \end{cases} \tag{3.12}$$

3.6 Statement: Initial Values for a Kinematic Wave (problem 02).

(Here we explore the issue of dissipation at shocks). In *Initial Values for a Kinematic Wave (problem 01)* we considered the following question: Solve the Kinematic Wave equation (for the conserved quantity u)

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = 0, \tag{3.13}$$

using the initial values

$$u(x,\,0) = \begin{cases} 0 & \text{for} \quad -\infty < x \leq -1, \\ 1+x & \text{for} \quad -1 \leq x \leq 0, \\ 1-x & \text{for} \quad 0 \leq x \leq 1, \\ 0 & \text{for} \quad 1 \leq x < \infty, \end{cases} \tag{3.14}$$

and introducing shocks in the solution (whenever needed to eliminate multiple values due to crossing of the characteristics). **Let the solution to this particular problem be** $\boxed{u = U(x,\,t)}$ — which is given explicitly in the answer to the *Initial Values for a Kinematic Wave (problem 01).* Then, **DO THE FOLLOWING:**

1. **VERIFY** directly (using the explicit form for U) that $A = \int_{-\infty}^{\infty} U(x,\,t)\,dx = 1$ for all times, so that the total amount of u is conserved, as it should.
2. **VERIFY** directly (using the explicit form for U) that the "energy" $E = \int_{-\infty}^{\infty} \frac{1}{2}\,U^2(x,\,t)\,dx$ is constant for $0 \leq t \leq 1$, and decreases for $t > 1$ — time derivative strictly less than zero. Since $t = 1$ is the time when a shock in the solution forms, this provides an explicit example showing that *shocks dissipate "energy" — even though (3.13) formally has no dissipation!* **The purpose of this problem is to understand a little of why and how this happens.**
3. Consider an arbitrary solution u to equation (3.13). **SHOW THAT,** as long as u has no shocks,[6] u also satisfies the equation

$$\left(\frac{1}{2}\,u^2\right)_t + \left(\frac{1}{3}\,u^3\right)_x = 0. \tag{3.15}$$

Thus, if $u \to 0$ as $|x| \to \infty$, the "energy" $E = \int_{-\infty}^{\infty} \frac{1}{2}\,u^2(x,\,t)\,dx$ will be a constant — this shows that the first part of the result in item **2** is generic.

[6]Thus u has derivatives, and satisfies the equation in the usual sense.

4. At a shock $x = x_s(t)$ for a solution u to equation (3.13), equation (3.15) does not hold. In fact, from the rules governing weak derivatives, it can be shown that[7]

$$\left(\frac{1}{2}\,u^2\right)_t + \left(\frac{1}{3}\,u^3\right)_x = \sum_{\text{shocks}} \left[-\frac{dx_s}{dt}\,\frac{1}{2}\,u^2 + \frac{1}{3}\,u^3\right]\,\delta(x - x_s(t)), \tag{3.16}$$

where $\delta(\cdot)$ is the Dirac delta function, and the brackets $[\cdot]$ denote the jump in the enclosed function across the shock — specifically: value immediately ahead of the shock minus value immediately behind. Thus, **SHOW THAT:**

$$\left(\frac{1}{2}\,u^2\right)_t + \left(\frac{1}{3}\,u^3\right)_x = \sum_{\text{shocks}} \frac{1}{12}\,[u]^3\;\delta(x - x_s(t)). \tag{3.17}$$

Since $[u] < 0$ at shocks for (3.13) — entropy condition **(SHOW THIS)**— the right hand side in (3.17) is negative, so that energy is dissipated. **INTEGRATE** this last equation from $x = -\infty$ to $x = \infty$, assuming that u vanishes as $|x| \to \infty$, and **OBTAIN AN EQUATION** for the time derivative of the energy E. **VERIFY** that E, as calculated in item **2**, satisfies this equation.

Note 1: Assume that (for each shock) along the curve $x = x_s(t)$ the function u has a discontinuity such that u, u_t, and u_x exist and are continuous on each side of the curve, and have left and right limits as the curve is approached. Furthermore, assume that dx_s/dt exists.

Note 2: Remember that the discontinuity across a shock must satisfy the Rankine-Hugoniot jump condition, as well as the entropy condition.

5. *How is it that the solutions to (3.13) end up with dissipation at shocks, when the equation itself has no explicit dissipation parameter?* The reason has to do with the fact that shocks arise in a singular limit as the dissipation parameter vanishes, as you will be asked to show here.

As explained in the lectures, adding shocks to the solutions of (3.13) — in order to resolve multiple values issues, as well as infinities in the derivatives — is not a mathematical step, but a (physical) *modeling issue*: the equation plus shock conditions includes further physical assumptions than the original model without them. Specifically: there is a diffusion-like process that "fights" (and stops) the steepening caused by the nonlinearity when the derivatives become large enough, stabilizing a transition in the solution from one value to another over

[7] You will be asked to show this in another problem.

a very thin layer. This layer is then modeled as being infinitely thin, with the solution being discontinuous across it: a shock.

For example, in the case of equation (3.13), a more complete model would include a small "viscosity" coefficient $0 < \nu \ll 1$, with the equation modified to

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \nu\,u_{x\,x}. \tag{3.18}$$

Then the solutions of (3.13) are obtained from the solutions of (3.18) in the limit $\nu \to 0$. As we saw in the lectures, for equations like this, the shocks can be modeled as traveling waves connecting two values of u over a layer of width $O(\nu)$ — i.e.: $u = F\left(\dfrac{x - s\,t}{\nu}\right)$, where s is the shock velocity and F is the shock structure function. The important conclusion from this that we need here is that:

$$\left.\begin{array}{l} \textit{Each shock layer has an x-width of size } O(\nu). \\ \textit{In the shock layer: } u = O(1),\ u_x = O(\nu^{-1}),\ u_t = O(\nu^{-1}), \text{ etc.} \end{array}\right\} \tag{3.19}$$

Consider now a solution of (3.18) that vanishes (fast enough) as $|x| \to \infty$ and **SHOW THAT**

$$\frac{dE}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}\,u^2(x,\,t)\,dx = -\nu\int_{-\infty}^{\infty} u_x^2(x,\,t)\,dx. \tag{3.20}$$

Hence E is decreasing. Furthermore[8] **ARGUE** that, as $\nu \to 0$, $\dfrac{dE}{dt}$ has a nonzero negative value if the solution has shocks — this limiting value of $\dfrac{dE}{dt}$ is, of course, the one that you were asked to derive in item **4**.

The results above illustrate how it is that the solutions to equation (3.13) dissipate energy, even though the dissipation parameter — the viscosity ν in (3.18) — vanishes. What happens is that (for small, but finite ν) *the amount of dissipation produced by each shock is (basically) independent of the value of* ν. This follows because the amount of dissipation is not just proportional to ν, but is also a function of the gradients involved — and the nonlinearity in the equation pushes these gradients up till they have size $O(\nu^{-1})$.

[8] Hint: use (3.19).

3.7 Statement: Infinite conservation laws for kinematic waves.

Consider the (kinematic wave) equation for some conserved scalar density $\rho = \rho(x,\, t)$ in one dimension

$$\rho_t + q_x = 0, \quad \text{where} \quad q = Q(\rho) \tag{3.21}$$

is the flow function — which we assume is sufficiently nice (say, it has a continuous first derivative). **SHOW THAT:** if $f = f(\rho)$ is some arbitrary (sufficiently nice) function, then there exists a function $g = g(\rho)$ such that *for any classical*[9] *solution of (3.21), we have*

$$f_t + g_x = 0. \tag{3.22}$$

In other words, (formally) f behaves as a "conserved" quantity, with flow function g. In particular:

$$\frac{d}{dt}\int_a^b f\,dx = g\big|_{x=a} - g\big|_{x=b} \tag{3.23}$$

expresses the "conservation" of f for any interval $a < x < b$.

EXTREMELY IMPORTANT: it is *crucial that the solution ρ have derivatives in the classical sense.* **When shocks are present, this result is false!** As you will be asked to show in another problem, when shocks are present (3.22) has to be replaced by

$$f_t + g_x = \sum_{\text{shocks}} c_s\,\delta(x - x_s(t)), \tag{3.24}$$

where $x = x_s(t)$ are the shock positions, δ is the Dirac delta function, and the $c_s = c_s(t)$ are some coefficients that are *generally not zero!* Thus the shocks act as sources (or sinks) for f.

3.8 Statement: Traffic Flow problem 01.

Determine the traffic density on a semi–infinity ($x > 0$) highway, for which the density at the entrance is

$$\rho(0,\, t) = \begin{cases} \rho_1 & \text{for } 0 < t < \tau, \\ \rho_0 & \text{for } \tau < t, \end{cases} \tag{3.25}$$

where $\tau > 0$ is constant, and the initial density is uniform along the highway — assume that $\rho(x,\, 0) = \rho_0$, for $x > 0$. Furthermore, assume that ρ_1 is lighter traffic than ρ_0, and that both are light traffic; in fact assume that $u(\rho) = u_{\mathsf{max}}\,(1 - \rho/\rho_{\mathsf{max}})$ and that $\rho_1 < \rho_0 < \rho_{\mathsf{max}}/2$. Sketch the density at various values of time.

[9] That is: no shocks, so no generalized derivatives are involved.

3.9 Statement: Traffic Flow problem 02.

In the simplest formulation of a traffic flow model, one starts with a car density $\rho(x,\,t)$ and a car flow rate $q = \rho\, u$, where $u = u(x,\,t)$ is the car velocity. Conservation of cars then yields the equation

$$\rho_t + q_x = 0.$$

The model is then closed by taking a quasi-equilibrium approximation, in which u is given as a function of ρ. Namely: $u = U(\rho)$, for some (decreasing) function U. This is motivated by the fact that (at steady state) people drive at a velocity that is directly correlated to how close the nearby cars are (i.e.: to ρ). Thus, *if the changes in ρ are not too fast (along car paths), the drivers have plenty of* **time** *(and* **space***) to adjust their velocity to the desired value given by the local density ρ* — hence then $u = U(\rho)$ is a good approximation.

(a) A more sophisticated theory of traffic flow assumes that the drivers do not move at a velocity determined (instantaneously) by the density. Instead, drivers accelerate in a manner so as to approach the desired velocity-density curve.

(a1) Formulate a model incorporating the simplest version of this idea.

(a2) What type of initial conditions are necessary to solve the new model?

Hint: *Assume that the drivers adjust their velocity to their local conditions, following a simple law in which they accelerate at a rate proportional to the difference between their actual velocity and the desired one — with an associated time constant $\tau > 0$. You should obtain, in this way, a model involving two equations, characterizing the evolution of $\rho = \rho(x,\,t)$ and $u = \rho(x,\,t)$.*

(b) *How good is your new model? Are there any hidden flaws in it that should be fixed?* In order to ascertain the answer to these questions **perform a (linearized) stability analysis of the steady state, equilibrium, solutions to your new model.** Namely:

(b1) Consider solutions of the form $\rho = \rho_0 + \delta\,\tilde{\rho}(x,\,t)$ and $u = u_0 + \delta\,\tilde{u}(x,\,t)$, with ρ_0 and u_0 constants, and $0 < \delta \ll 1$ infinitesimal. **Derive equations for $\tilde{\rho}$ and $\tilde{u}$.**

(b2) Find the normal mode solutions to the equations derived in item **b1**. That is, look for solutions of the form $\tilde{\rho} = \mathrm{Re}(a_1\, e^{i\,k\,x+\sigma\,t})$ and $\tilde{u} = \mathrm{Re}(a_2\, e^{i\,k\,x+\sigma\,t})$, where a_1, a_2, k and σ are constants, with k real. Then **find a formula characterizing** σ as a function of the wave number $-\infty < k < \infty$. For every k, there are two possible choices for σ.

(b3) Show that *the equilibrium solutions $\rho = \rho_0 =$ const. and $u = u_0 =$ const. are unstable to infinitesimal perturbations of any wave-length.* Namely, show that for every k, there is a choice of the σ in item **b2** — say $\sigma = \sigma_2$ — with $\mathrm{Re}(\sigma_2) > 0$.

(b4) Show that *the growth rate for the unstable modes in item* **b3** *can be arbitrarily large.* In fact, show that $\mathrm{Re}(\sigma_2) = O(\sqrt{|k|})$ for $|k| \gg 1$. This shows that *the (linear) equations for $\tilde{\rho}$ and $\tilde{u}$ derived in item* **b1** *are* **ill-posed!** The equilibrium solutions are not just unstable to infinitesimal perturbations of any wavelength, they are catastrophically and violently unstable $\longleftarrow$ spells **DISASTER for the model in item a.**

Note: one way in which this could avoid being a disaster is if, somehow, the nonlinearities in the problem were to (eventually) intervene, clipping the growth and stabilizing the solution to some non-equilibrium, but "nice and smooth" solution of the equations. In the next item we show that this is not possible.

(c) Optional/challenge. You can safely skip to item **d** if you cannot (or do not want to) do this. The purpose here it to refine the results in item **b**, and show that it is not only the equilibrium solutions of the model in item **a** that are violently unstable. You will be asked to **show that:** *Any and all smooth solutions — $\rho = \rho_0(x,\, t)$ and $u = u_0(x,\, t)$ — to the model in item* **a** *are violently unstable to high frequency perturbations, with growth rates that go to infinity as the wavelength vanishes.* Proceed as follows:

(c1) Look for solutions of the form $\rho = \rho_0(x,\, t) + \delta\, \tilde{\rho}(x,\, t)$ and $u = u_0(x,\, t) + \delta\, \tilde{u}(x,\, t)$, where $0 < \delta \ll 1$ is infinitesimal and $(\rho_0,\, u_0)$ is a smooth solution to your new model in item **a**. **Derive equations for $\tilde{\rho}$ and $\tilde{u}$.** In the next item you will be asked to show that *these equations are ill-posed.*

(c2) The equations derived in item **b1** are linear and constant coefficients — hence a normal mode analysis is easy to perform using exponentials. The equations derived in item **c1**, on the other hand, are linear — but with variable coefficients. Doing a complete normal mode study analytically is not possible. On the other hand, the interest here is only with computing the growth rate for high frequency perturbations of the smooth solution $(\rho_0,\, u_0)$. Hence, a W.K.B.J. type of approximation is possible. Thus, **seek**

solutions for the (linear) equations derived in item c1, of the form:

$$\left.\begin{aligned}\tilde{\rho} &= \mathrm{Re}\left(\overline{\rho}(x,\,t,\,T;\,\epsilon)\,\exp\left(\frac{i}{\epsilon}\,\theta(x,\,t)\right)\right),\\ \tilde{u} &= \mathrm{Re}\left(\overline{u}(x,\,t,\,T;\,\epsilon)\,\exp\left(\frac{i}{\epsilon}\,\theta(x,\,t)\right)\right),\end{aligned}\right\} \quad \text{where } 0<\epsilon\ll 1,\;\; T=\frac{t}{\sqrt{\epsilon}}, \tag{3.26}$$

ϵ is a measure of the wavelength, and both $\overline{\rho}$ and $\overline{u}$ have expansions in powers of $\sqrt{\epsilon}$.

Motivation: In any sufficiently small neighborhood in space time, the variations in the coefficients of the equations derived in item **c1** can be neglected. A high frequency wave will thus see the equations as "constant coefficients" in each small region of space-time, and so should behave (locally) like a plane wave. This is what the form in (3.26) is designed to provide, since near any fixed $(x_0,\,t_0)$ it yields

$$\begin{aligned}\tilde{\rho} &\approx \mathrm{Re}\left(\overline{\rho}(x_0,\,t_0,\,T;\,\epsilon)\,\exp\left(\frac{i}{\epsilon}\,\theta_0 + i\,k_0\,(x-x_0) - i\,\omega_0\,(t-t_0)\right)\right),\\ \tilde{u} &\approx \mathrm{Re}\left(\overline{u}(x_0,\,t_0,\,T;\,\epsilon)\,\exp\left(\frac{i}{\epsilon}\,\theta_0 + i\,k_0\,(x-x_0) - i\,\omega_0\,(t-t_0)\right)\right),\end{aligned}$$

where $\boxed{k=\theta_x,\;\;\omega=-\theta_t,}$ and the subscript zero indicates evaluation at $(x,\,t)=(x_0,\,t_0)$.

Question: why is it necessary to add a dependence on $T/\sqrt{\epsilon}$ above in (3.26)? Why do we need to introduce square roots of ϵ into the solution?

(c3) You can further confirm the inability of the nonlinear terms (in the new model derived in item **a**) to fix the arbitrarily large growth rates that occur with high frequency perturbations (as the wavelength becomes smaller) to smooth solutions, by asking (and answering) the question: *At what amplitude do nonlinear effects become important in the evolution of a high frequency solution, such as the ones modeled by the W.K.B.J. type ansatz (3.26) in the linear regime?* **Answer the question, and show that:** the nonlinear terms do not become important as long as the amplitude of the perturbations remains small. This means that perturbations of arbitrarily large growth rates for arbitrarily short wavelengths can grow to finite amplitude without being impaired by the nonlinear terms. This spells disaster for the **model in item a, which this shows to be ill-posed.**

Hint: To answer the question you must move away both from the δ infinitesimal assumption, and from the W.K.B.J. use of exponentials to describe solutions. Thus use an ansatz of the form

$$\left.\begin{array}{rcl}\rho &=& \rho_0 + \qquad \delta\,\tilde{\rho}(\Psi,\,T,\,x,\,t\,;\,\epsilon,\,\delta),\\ u &=& u_0 + \sqrt{\epsilon}\,\delta\,\tilde{u}(\Psi,\,T,\,x,\,t\,;\,\epsilon,\,\delta),\end{array}\right\} \tag{3.27}$$

where (1) $\Psi = \dfrac{1}{\epsilon}\,\theta(x,\,t)$ — with θ the same as in (3.26). Further, let $k = \theta_x$, as before.

(2) $0 < \delta,\, \epsilon \ll 1$ — neither being infinitesimal. Here ϵ is a measure of the wavelength and δ is a measure of the nonlinearity.

(3) $\tilde{\rho}$ and $\tilde{u}$ have appropriate expansions in powers of ϵ and δ, starting at $O(1)$.

(4) $u_0 = u_0(x,\,t)$ and $\rho_0 = \rho_0(x,\,t)$ are a smooth solution to the model in item **a**.

Then show that the nonlinearities are never important at leading order, as long as they are small (i.e.: $0 < \delta \ll 1$).

Note: the reason why the perturbation to u_0 in (3.27) has to be scaled by an extra smallness factor $\sqrt{\epsilon}$ should become clear to you from your answer to item **c2**.

(d) The model of item **(a)** essentially introduces a delay (due to the time a driver takes to accelerate) in a driver's response to the observed density. Unless the drivers look far enough ahead to compensate for this response time, this delay process leads to an extremely bad instability to high frequency perturbations, as the cars systematically accelerate towards a velocity that is the wrong one when achieved.

Implement the simplest modification of the model in item (a), incorporating the fact that the drivers take preventive action to account for the delay.

Hint. A simple way to put it is that: *the drivers accelerate towards a velocity that matches the conditions they expect to find once the acceleration process ends.* In other words, their target velocity is not $u_{\mathsf{target}} = U(\rho)$, but some corrected velocity, where the correction arises from what they see the conditions (density) ahead of them are. Think of a simple way to write $u_{\mathsf{target}} = U(\rho)+$ correction, where the correction involves only ρ, ρ_x, and a parameter $\nu > 0$ with the dimensions of a diffusion (length square over time).

(e) **Repeat the analysis in item b, for the corrected model in item d. Show that, if ν is taken large enough, the uniform equilibrium solutions can be made stable.** Incorporating the "look-ahead" behavior by the driver eliminates the bad behavior, and a well behaved mathematical model is thus obtained.

3.10 Statement: Envelopes and cusps.

Consider the problem

$$c_t + c\,c_x = 0, \quad \text{where} \quad c(x,\,0) = \mathcal{C}(x), \;\; \text{for} \;\; -\infty < x < \infty. \tag{3.28}$$

e1. Write the characteristic curves for this problem, each one parameterized by time, and labeled by the value of $x = s$ at time $t = 0$. That is, write formulas for the characteristics of the form $x = X(s,\,t)$.

e2. Write the equation for the envelope of the characteristics, and express the envelope in parametric form $x = X_e(s)$ and $t = T_e(s)$.

e3. Assume that $\mathcal{C}$ has an inflection point at $x = 0$. In fact, assume that

$$\mathcal{C}(0) = 0, \;\; \mathcal{C}'(0) = -a < 0, \;\; \mathcal{C}''(0) = 0, \quad \text{and} \quad \mathcal{C}'''(0) = 2\,a\,b > 0, \tag{3.29}$$

where the primes denote derivatives.

Let $x_c = X_e(0)$ and $t_c = T_e(0)$. **Show that T_e has a local minimum at $s = 0$, and that the envelope has a cusp at $(x_c,\,t_c)$.**
HINT: *Expand the equations for s small.*

e4. Replace (3.28) by

$$c_t + c\,c_x = -c, \quad \text{where} \quad c(x,\,0) = \mathcal{C}(x), \;\; \text{for} \;\; -\infty < x < \infty. \tag{3.30}$$

Repeat steps e1 and e2 for this problem. Question: what condition is needed on $\mathcal{C}'$ so that the characteristics of (3.30) actually HAVE an envelope in the upper half space-time plane? That is, so that $T_e(s) > 0$ somewhere.

4 Hamilton Jacobi and Eikonal Problems.

4.1 Statement: Eikonal equation (problem 01).

Consider the Eikonal equation (for the wave equation in 2-D) in a context where the wave speed is a constant (homogeneous media), so that we can set (upon non-dimensionalization) $c = 1$. Then

$$\phi_x^2 + \phi_y^2 = 1. \tag{4.1}$$

Consider now the situation where the wave-front $\phi = 0$ is a parabola. Specifically:

$$\phi = 0 \quad \text{on} \quad y = x^2, \tag{4.2}$$

with propagation direction towards y increasing. For this problem, **this is what you should do:**

1. Find the family of all the rays (characteristics) for $t > 0$. The easiest way is to describe it is parametrically: $x = x(s,\, t)$ and $y = y(s,\, t)$, where $x(s,\, 0) = s$, $y(s,\, 0) = s^2$, and t is time of travel along the ray (for the wave-fronts) starting from the initial wave-front — i.e.: $\phi = t$.

2. Find the caustic. The caustic is the envelope of the family of rays = the locus of the intersections of infinitely close neighbors in the family of rays[10] = a curve such that each point in it belongs to one of the rays, and it is tangent to the ray there.[11] In this case of constant wave speed, where the rays are straight lines, the caustic is also the locus of the centers of curvature of the wave-fronts (all the wave-fronts have the same set of centers of curvature).

From the parametric description of the family of rays in item **1**, you should be able to obtain the caustic parametrically in terms of s. However, you should also be able to find a very simple formula — of the form $(y - y_a)^\alpha = \text{const.}\,(x - x_a)^2$ — for the caustic. Do so.

3. Do a sketch of the wave-front $\phi = 0$, and of the caustic. Indicate the region of the plane where the rays cross and give rise to multiple values in the solution to the equation.

4. The earliest time at which a ray crossing occurs corresponds to the singular point in the caustic (the arête). Find the position of the arête in space, the ray, and the time (or wave-front, as $\phi = t$) it correspond to. Let these parameters be x_a, y_a, s_a, and t_a. Explicitly show that t_a is the earliest time at which a crossing of rays occurs.

5. Add to the sketch in item **3** the wave-front $\phi = t_a$. This wave-front is singular at the arête; describe the nature of this singularity. In particular, show that the wave-front satisfies (at leading order) a formula of the form $(y - y_a) \sim \text{const.}\,(x - x_a)^\mu$ near the arête.

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[10] $(x_*,\, y_*) \in$ caustic $\iff$ $x_* = x(s,\, t) = x(s + ds,\, t + dt)$ and $y_* = y(s,\, t) = y(s + ds,\, t + dt)$, for some s and t.

[11] $(x_*,\, y_*) \in$ caustic $\iff$ for some s and t: $x_* = x(s,\, t)$, $y_* = y(s,\, t)$, and $(x_t(s,\, t),\, y_t(s,\, t))$ is tangent to the caustic at $(x_*,\, y_*)$.

4.2 Statement: Eikonal equation (problem 02).

Consider the Eikonal equation (for the wave equation in 2-D) in the context where the wave speed is not a constant (non-homogeneous media). In particular, consider the following situation

$$c^2\,(\phi_x^2 + \phi_y^2) = 1, \tag{4.3}$$

where $c = c_1 > 0$ for $y > 0$, and $c = c_2 > 0$ for $y < 0$, with $c_1 \neq c_2$. Of course, in a situation like this, we must worry about what is the meaning of the equation for $y = 0$. From the derivation in the lectures, where we saw that ϕ is a phase, it is easy to see that what we want to require in that ϕ be continuous across $y = 0$, with the equation satisfied on each side. On the other hand, *the gradient of ϕ will, most definitely, not be continuous.* In fact, investigating what happens with $\nabla\phi$ across $y = 0$ is the purpose of this problem.

Assume that $\nabla\phi$ is continuous on each side of $y = 0$, with continuous limits on each side as $y \to 0$. Let $\nabla^+\phi = \lim\limits_{y\to 0,\; y>0} \nabla\phi$ and $\nabla^-\phi = \lim\limits_{y\to 0,\; y<0} \nabla\phi$. **Find a relationship between $\nabla^+\phi$ and $\nabla^-\phi$.**

Note: since $\nabla\phi$ is the direction of the rays — given by $\dfrac{d\vec{r}}{dt} = c^2\,\nabla\phi$ — the relationship you find should be equivalent to Snell's law. **Show** that this is, indeed, the case.

5 HyperbolicEquations.

5.1 Statement: The importance of being hyperbolic.

Consider initial value problems of the form

$$\vec{u}_t + \mathcal{A}\,\vec{u}_x = \mathcal{B}\,\vec{u}, \quad \text{with } \vec{u} = \vec{u}_0(x), \tag{5.1}$$

where $\mathcal{A}$ and $\mathcal{B}$ are $N \times N$ square, constant, real matrices, and $\vec{u} = \vec{u}(x,\,t)$ is a column N vector real valued function. The purpose of this exercise is to *investigate general conditions under which this problem is either well posed or ill posed.*

(a) Show that *if $\mathcal{A}$ has an eigenvalue λ such that $Im(\lambda) \neq 0$, then (5.1) is ill posed.*

Hint a1: Look for normal mode solutions of the form $\vec{v}\,e^{i\,k\,x+\sigma\,t}$, where $\vec{v}$ is a (constant) vector, $-\infty < k < \infty$, and σ is some constant. Show that solutions of this type can be found with arbitrarily large growth rates. Specifically, show that Re$(\sigma) \to \infty$ as $|k| \to \infty$.

Hint a2: If $\mathcal{C}$ is close to $\mathcal{A}$, then the eigenvalues of $\mathcal{C}$ are close to the eigenvalues of $\mathcal{A}$.

(b) Show that *if (5.1) is strictly hyperbolic,*[12] *then the growth rate for the normal mode solutions is bounded, no matter what the choice of $\mathcal{B}$ is.* This can be parlayed into a proof that (5.1) is well posed, by showing that the normal mode solutions form a complete set (you are not being asked to do this).

Hint b1: Look for normal mode solutions of the form $\vec{v}\, e^{i\,k\,x+\sigma\,t}$, where $\vec{v}$ is a (constant) vector, $-\infty < k < \infty$, and σ is some constant. Express σ as the eigenvalue in an appropriate eigenvalue problem. From this you will be able to show that σ is a continuous function of k — see Hint a2. Thus, on any bounded set of k's, Re(σ) is bounded. Investigate now the behavior of σ as $|k| \to \infty$, and show that (in this limit) Re(σ) remains bounded.

Hint b2: In this case the eigenvalues and eigenvectors of the matrix $\mathcal{A} + \epsilon\,\mathcal{C}$, where $0 < \epsilon \ll 1$ and $\mathcal{C}$ is some constant matrix, can be calculated using a (convergent) power series in ϵ expansion.

When the eigenvalues of $\mathcal{A}$ are real, but have multiplicities greater than one, the analysis becomes a little more complicated. To keep the analysis simple, **assume that** $N = 2$ **for parts c and d below.**

(c) Show that *if (5.1) is hyperbolic, but not strictly hyperbolic,*[13] *then it is well posed.* This is true for any N, but you are being asked to show it only for $N = 2$.

Hint c1: Show that in this case $\mathcal{A}$ is a multiple of the identity matrix. Then reduce the problem to a system of o.d.e.

(d) Show that *if $\mathcal{A}$ has real eigenvalues, but it is not diagonalizable, then $\mathcal{B}$ can be selected so that (5.1) is ill posed.* Note that, in this case (5.1) is **not hyperbolic.**

Hint d1: Take

$$\mathcal{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \tag{5.2}$$

where λ is some constant, and investigate the normal mode solutions. Show that, as long as $\mathcal{B}_{21} \neq 0$, there are normal modes with a growth rate that goes to infinity as $|k| \to \infty$.

[12] That is: $\mathcal{A}$ is real diagonalizable, and the eigenvalues are all distinct.

[13] That is, $\mathcal{A}$ is real diagonalizable, but the eigenvalues are not distinct.

6 Point Sources and Green functions.

6.1 Statement: Green's functions for the wave equation.

In this problem we consider the initial value problem for the wave equation — with constant wave speed c — in R^d, and find explicit integral formulas for its solution.
Without loss of generality we can assume $c = 1$. **The problem is** then

$$u_{tt} = \Delta u, \quad \text{for} \quad -\infty < x_j < \infty \; (1 \leq j \leq d) \quad \text{and} \quad t > 0, \tag{6.1}$$

where $u(\vec{x},\, 0) = u_0(\vec{x})$, $u_t(\vec{x},\, 0) = v_0(\vec{x})$, $\vec{x} = (x_1,\, x_2,\, \dots x_d) \in R^d$, $\Delta = \sum \partial^2_{x_j}$, and $u = u(\vec{x},\, t)$ is a real valued (scalar) function to be found. To solve this problem, we: **(1-st)** Seek (explicit) special solutions (Green's functions) to the wave equation — $u = G_1(\vec{x},\, t)$ and $u = G_2(\vec{x},\, t)$ — satisfying

$$G_1(\vec{x},\, 0) = \delta(\vec{x}) \qquad \text{and} \quad \partial_t G_1(\vec{x},\, 0) = 0, \tag{6.2}$$

$$G_2(\vec{x},\, 0) = 0 \qquad\quad \text{and} \quad \partial_t G_2(\vec{x},\, 0) = \delta(\vec{x}), \tag{6.3}$$

where $\delta(\vec{x}) = \delta(x_1)\,\delta(x_2)\dots\delta(x_d)$ is the Dirac delta function in d dimensions. **(2-nd)** Use these solutions to produce integral formulas for the solution of (6.1).

Remark 6.1 *Equation (6.1) is linear hyperbolic, with a single propagation speed $c = 1$ in all directions. Since the initial conditions involve non-trivial data at $\vec{x} = 0$ only, we expect that:*
6.1-a. G_1 **and** G_2 **vanish identically for** $r > t$ *— where $r = \sqrt{\sum x_j^2}$.*
6.1-b. G_1 **and** G_2 **are singular for** $r = t$ **only.** *This means that the only place where G_1 or G_2 may behave in a non-classical way (see warning below) is at $r = t$. Everywhere else G_1 and G_2 should satisfy the equation in the classical (strong) sense.*
6.1-c. Warning: *Neither G_1 nor G_2 is a function in the classical sense. Both are "generalized" functions, and satisfy the equation in the weak sense* **only.** *Keep this in mind when doing the problem:* any equation you write must make sense.[14] *See remarks **6.2** and **6.3** for some helpful (I hope) facts.*

Below are the **TASKS FOR THIS PROBLEM:**

[14] This does not mean that you are expected to provide a rigorous (in the strict mathematical sense) answer. On the other hand, writing equations that have no meaning, or making false arguments is not allowed.

The stuff below "guides" you through the construction of the solution to the wave equation initial value problem in any number of dimensions. It does not follow the usual path that you may find in books. I want you to **do it in the way requested below,** *for at least three reasons:*
(1) For practice with weak solutions and generalized functions.
(2) Approach less tied up to the peculiar properties of the equation.
(3) No point in assigning something that you can find in any book.

01. Show that, given G_1, one can take $G_2(\vec{x},\, t) = \int_0^t G_1(\vec{x},\, s)\, ds$. Vice versa, given G_2, one can take $G_1 = \partial_t G_2$. Thus, once either of G_1 or G_2 is known, the other one follows easily.

02. Argue that G_1 and G_2 should have the form $G_1 = t^{-d}\, U(z)$ and $G_2 = t^{-d+1}\, V(z)$, for some functions U and V, where $z = r/t$ and $r = \sqrt{\sum x_j^2}$. Then, from item **01** it follows that $U = (1-d)\, V - z\, V'$ — where $' = \dfrac{d}{dz}$, and $V = \int_0^1 U(z/s)\, s^{-d}\, ds$.
Hint 02: *Show that the problem defining these functions is invariant under rotations and appropriate scalings of the variables. You will need some of the results in remark* ***6.2*** *here.*

03. (A) Use (6.1), **6.1-a**, and $G_1 = t^{-d}\, U(z)$ to **obtain a FIRST order linear O.D.E. for** U.
Hint 03a: *Do not expand the expression for the Laplacian of a rotationally symmetric function in your calculations — namely, use:* $\Delta = r^{1-d}\,\partial_r\, r^{d-1}\,\partial_r$. *You should be able to write a somewhat similar formula for* $\partial_t^2 G_1$. *Using these you will be able to integrate* once *the second order O.D.E. for* U *that direct substitution produces.* **Warning:** see remark **6.3**.

(B) Show that odd dimensions are special. If $d = 2\,m + 1$, the equation for U in part **A** can be integrated once more, to arrive at what is (essentially) an algebraic equation for U.
Hint 03b: *The O.D.E. for* U *obtained in part* **A** *is first order. Hence, it always has an integrating factor. However, when* d *is even, the integrating factor has a singularity at the wrong place (*$z = 1$*), that makes multiplication of* U *by it have a not-too-clear meaning. For* d *odd, the integrating factor is smooth, and multiplication by it does not create ambiguities — on the other hand, when interpreting the equation that results upon integration, be mindful of example* **6.3-a** *in remark* ***6.3!***

(C) Show that if $S = S(z)$ is a solution of the equation for U in part **A** for $d = d_1$ space dimensions, then $\dfrac{1}{z}\,S'$ is a solution of the corresponding equation for $d = d_1 + 2$.

04. Find the form that the initial conditions (6.2) take in terms of U. For the values of the constants that will appear in these conditions, you will need to do item **09**.

Hint 04: *Let $\phi = \phi(\vec{x})$ be an "arbitrary" test function, then (6.2) says that*

$$\phi(0) = \lim_{t\to 0} \int G_1(\vec{x},\, t)\, \phi(\vec{x})\, dx_1 \dots dx_d \quad \text{and} \quad 0 = \lim_{t\to 0} \int \partial_t G_1(\vec{x},\, t)\, \phi(\vec{x})\, dx_1 \dots dx_d. \tag{6.4}$$

Furthermore, since $G_1 = G_1(r,\, t)$, it is enough to consider test functions of the form $\phi = \phi(r)$. However, note that ϕ must be smooth as a function in R^d, thus the odd derivatives of ϕ must vanish at the origin: $0 = \phi'(0) = \phi'''(0) = \dots$

Note that, because U vanishes for $z > 1$, you should be able to show that the **second condition in (6.4) — the one involving $\partial_t G_1$ — is always satisfied.** Showing this will simplify (a little) parts **05-07** below. However, you are not required to show this. If, on the other hand, you want a (tinny) challenge, here is a **tip:** Generalized functions, such as U, are defined as continuous, linear functionals[15] $\mathcal{L}$ on the set of all test functions. Continuity means the usual: $\lim_{\phi\to\Phi} \mathcal{L}(\phi) = \mathcal{L}(\Phi)$, where (for test functions) $\phi \to \Phi$ means (uniform) convergence of not just the function, but of each of the derivatives as well.

05. For $d = 1$ **find** U **(thus** G_1**), and then** G_2 (using the results of item **01**). Use then G_1 and G_2 to **write an integral formula for the general solution to the initial value problem for the wave equation** in (6.1), with $u(x,\, 0) = u_0(x)$ and $u_t(x,\, 0) = v_0(x)$.

Hint 05: *The wave equation (6.1) is linear (thus sums and integrals over solutions are solutions), invariant under translation, and one can write $u_0(x) = \int \delta(x-s)\, u_0(s)\, ds$ and $v_0(x) = \int \delta(x-s)\, v_0(s)\, ds$.* Further stuff that could prove useful *(here and in parts* **06-07***): (i) Item* **03-B***; (ii) Remark **6.3**; (iii) The formulas $c\,\delta(c\,x) = \delta(x)$ and $c^2\,\delta'(c\,x) = \delta'(x)$ for any constant $c > 0$.*

06. Same as in **05**, but for $d = 3$.

Hint 06: *When writing the formula for the general solution of the initial value problem, use the fact that $G_1 = \partial_t\, G_2$. This will make them look a lot simpler.*

[15] The value of the functional is interpreted as the integral of the generalized function times the test function.

07. Same as in **05**, but for $d = 2$.

Hint 07a: This case is a little tricky! *The equation for* U *that you obtained in item* **03** *could lead you to an expression for* U *that has no clear meaning!* The problem is *that for a function* $f = f(z)$ *to make sense as a generalized function (and thus have derivatives in this sense, even if non-differentiable in the classical sense), it must be integrable — so that expressions such as* $\int f(z)\,\phi(z)\,dz$ *are defined for test functions.* In order to get around this difficulty, *write* $U(z) = S'(z)$ *as a derivative (with* S *vanishing for* $z > 1$*). Then, without doing any "illegal" operations*[16] *you should be able to integrate the equation (once) and obtain a 1-st order O.D.E. for* S*. The solution to this equation will also be singular, but integrable.*

The suggestion in the prior paragraph will not completely pull you out of dangerous waters! There is still one more trick: *Because* S *has a singularity (where it is not differentiable in the classical sense), to show that it does (indeed) satisfy the O.D.E., you will have to interpret the meaning of the O.D.E. in terms of test functions.*

Hint 07b: *Once you obtain* $U = U(z)$ *you will have to translate it into a formula for* G_1 *— which is a function of* r *and* t*. When you do this, translate derivatives with respect to* z *in derivatives with respect to* t *— not* r*. This will make writing the formulas for the general solution to the initial value problem a lot simpler — the same tip given in hint* **06**.

08. Dimension reduction.

Let $G_1 = G_1(x_1,\, x_2 \dots x_n,\, t)$ be the solution to the problem in (6.1 – 6.2) in $d = n$ dimensions.

Show that:

$$G = \int_{-\infty}^{\infty} G_1(x_1,\, x_2 \dots x_n,\, t)\, dx_n \tag{6.5}$$

solves the problem in (6.1 – 6.2) in $d = n - 1$ dimensions. Use this to **verify your answer in item 07** — i.e.: obtain the G_1 for $d = 2$ from the G_1 for $d = 3$.

09. Prove the results in remark **6.2** below.

Hint 09: *The formulas involve equalities between generalized functions. Hence, integrate each side versus a test function, and show that they give the same answer.*

10. Do the calculations in remark **6.4** below.

Hint 10: *Do the easy cases* $d = 2,\ 3$ *first, to see how it all works.*

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[16] For example: multiplication by something that is singular at $z = 1$ is illegal.

Remark 6.2 (A few useful properties of the Dirac delta function $\delta = \delta(x)$ — x a real variable).

6.2-a. *Let $g = g(x)$ be a continuous function, differentiable (with a non-zero derivative) near every point where it vanishes. Then*

$$\delta(g(x)) = \sum \frac{1}{|g'(x_n)|}\,\delta(x - x_n),$$

where the sum is over all x_n such that $g(x_n) = 0$.

Examples: $\boxed{|c|\,\delta(c\,x) = \delta(x)}$ *for any constant $c \neq 0$, and* $\boxed{\delta(|x| - 1) = \delta(x-1) + \delta(x+1).}$

6.2-b. *Let $f = f(x)$ have enough (continuous) derivatives at the origin. Then*

$$f(x)\,\delta^{(n)}(x) = f_0^0\,\delta^{(n)}(x) - B_1^n\,f_0^1\,\delta^{(n-1)}(x) + B_2^n\,f_0^2\,\delta^{(n-2)}(x) - \ldots + (-1)^n\,f_0^n\,\delta^{(0)}(x),$$

where f_0^j is the value of the j-th derivative of f at $x = 0$, and $B_j^n = \dfrac{n!}{(n-j)!\,j!}$ are the combinatorial coefficients.

Examples

$$\begin{aligned}
f(x)\,\delta^{(0)}(x) &= f^{(0)}(0)\,\delta^{(0)}(x) && , \\
f(x)\,\delta^{(1)}(x) &= f^{(0)}(0)\,\delta^{(1)}(x) - f^{(1)}(0)\,\delta^{(0)}(x) && , \\
f(x)\,\delta^{(2)}(x) &= f^{(0)}(0)\,\delta^{(2)}(x) - 2\,f^{(1)}(0)\,\delta^{(1)}(x) + f^{(2)}(0)\,\delta^{(0)}(x) && , \\
f(x)\,\delta^{(3)}(x) &= f^{(0)}(0)\,\delta^{(3)}(x) - 3\,f^{(1)}(0)\,\delta^{(2)}(x) + 3\,f^{(2)}(0)\,\delta^{(1)}(x) - f^{(3)}(0)\,\delta^{(0)}(x).
\end{aligned}$$

Remark 6.3 (Operating with generalized functions).

The wave equation (6.1) is linear, so that no products of generalized functions (which can be meaningless) should appear in your calculations. On the other hand, linear combinations of U and its derivatives (all generalized functions), with coefficients that are functions (of z and t), will show up. In this situation, as long as the coefficients are not singular (or have zeros), you can operate with generalized functions just as if they were regular "nice" functions. However: beware of situations were the coefficients have singularities (or zeros) where the generalized functions fail to be "normal" functions *(here for $z = 1$). The examples below illustrate the type of problems you may face:*

Example 6.3-a. *This shows the type of problem that a zero causes: $x^2\,\delta(x) = 0$, but $\delta \neq 0$ — where this means that $\int \delta(x)\,\phi(x)\,dx$ does not vanish for all test functions.*[17] *By contrast, if you knew that $x^2\,f(x) = 0$ for some "classical" function f, you would be able to immediately conclude that $\int f(x)\,\phi(x)\,dx = 0$ for* any *test function, so that (effectively) $f = 0$.* In general:

from $x\,M = 0$ you can conclude that M is proportional to $\delta(x)$,

[17] In fact, it does not vanish for any test function such that $\phi(0) \neq 0$.

from $x^2\,M = 0$ you can conclude that M is a linear combination of $\delta(x)$ and $\delta'(x)$,

from $x^3\,M = 0$ you can conclude that M is a linear combination of $\delta(x)$, $\delta'(x)$, and $\delta''(x)$,

and so on.

Example 6.3-b. *If M is a generalized function — singular at $x = 0$ — then $x^2\,M$ makes sense (because x^2 is smooth at 0), but $x^{1/3}\,M$, or $(\log|x|)\,M$ or $x^{5/2}\,M$ may not — e.g. what is the meaning of $x^{1/3}\,\delta'(x)$, or $(\log|x|)\,\delta(x)$, or $G = x^{5/2}\,\delta'''(x)$? On the other hand, $x^{2/3}\,\delta(x)$ is perfectly OK — in fact $x^{2/3}\,\delta(x) = 0$.*

Example 6.3-c. *$(x^{2/3}\,\delta(x))' = 0$, but you cannot use the product rule, since neither $x^{2/3}\,\delta'(x)$, nor $x^{-1/3}\,\delta(x)$ have a clear meaning.*

Finally: *If a generalized function $G = G(x)$ is known to vanish everywhere but at some point*[18] *x_0, then — for any test function $\phi = \phi(x)$ — the value of $\int G(x)\,\phi(x)\,dx$ can only involve a linear combination of the values $\phi(x_0)$, $\phi'(x_0)$, $\phi''(x_0)$, etc. In other words, G must be a linear combination of $\delta(x - x_0)$, $\delta'(x - x_0)$, $\delta''(x - x_0)$, etc.*

Remark 6.4 Areas and volumes.

Let A_{d-1} be the "area" of the unit sphere S^{d-1} in R^d. You can calculate A_{d-1} as follows:

6.4-a. *Let V_d be the "volume" of the unit ball B^d in R^d.*

6.4-b. *Decompose B^d into "slices" perpendicular to some fixed diameter. Each slice is a "ball" in R^{d-1}. Integrate to get an expression relating V_d to V_{d-1}, and then get V_d by induction.*

6.4-c. *Decompose B^d into concentric spherical shells. Integrate and get an expression relating V_d to A_{d-1}.*

Remark 6.5 Initial value problem for the wave equation is well posed.

In this problem no assumption about the existence, or uniqueness, of the solutions to the wave equation is made. The objective is to construct special solutions G_1 and G_2 with the various properties stated earlier. *However: once we have G_1 and G_2, never mind how we found them, the formulas in items* **05, 06**, *and* **07** *(as well as those for $d > 3$) show existence by explicitly providing a solution to the initial value problem. This solution has all the expected properties: (i) Continuous dependence on the initial data; (ii) Propagation of singularities following the characteristics; (iii) Solution depends only on the initial data within its domain of dependence; etc.*

[18] For example, you know that $f(x)\,G(x) = 0$ for some smooth function f with a zero at x_0.

Thus, in order to obtain the result that the initial value problem for the wave equation is well posed, the only missing element is uniqueness. As it turns out, uniqueness has a rather simple proof: the answers will have a sketch of the proof.

6.2 Statement: Cerenkov radiation and Mach cone.

Cerenkov radiation is the electromagnetic radiation emitted when a charged particle passes through an insulator at speed greater than the light speed in the medium. It shows up as a glowing blue cone of light with the traveling particle at its tip. It is somewhat analogous to the sonic boom produced by a supersonic aircraft. Below a simple model that captures the essence of the phenomena.

Consider a point traveling along a straight line at speed v, forcing the wave equation in 3-D (for an homogeneous and isotropic media). Assume also that $\mathrm{v} > c$, where c is the wave speed. In appropriate non-dimensional units, the mathematical problem is

$$u_{tt} - \Delta\, u = \delta(x)\, \delta(y)\, \delta(z - \beta\, t), \tag{6.6}$$

where $\beta = \mathrm{v}/c > 1$ and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplace operator in 3-D. We are *interested in the solution of this problem when the point moves into a media at rest,*[19] *in unbounded space,*[20] *and all the transients are gone* — i.e.: the motion started far in the past.

A. Show that the problem can be reduced to an initial value problem for the wave equation in 2-D, and use this to find the solution. What is the half angle of the Cerenkov cone?

B. What if $\boxed{0 \leq \beta < 1?}$ Can the problem be reduced to a 2-D initial value problem? Why not?

HINT-1. (An 18.03 hint!) Example: how to reduce to an initial value problem the impulse problem for the harmonic oscillator $\ddot{u} + u = c\,\delta(t)$, with $u \equiv 0$ for $t < 0$. Look for a continuous solution where u solves the homogeneous problem for $t \neq 0$, and $\dot{u}$ has an appropriate jump at $t = 0$. Hence, for $t > 0$, u solves the homogeneous problem, with initial conditions $u(0) = 0$ and $\dot{u}(0) = c$.

HINT-2. In order to solve the initial value problem for the wave equation in 2-D, you will need the Green's functions for the equation. These are:

$$G_1 = \frac{1}{2\,\pi}\,\frac{\partial}{\partial t}\left(\frac{H(t-r)}{\sqrt{t^2 - r^2}}\right),$$

[19]That is: $u \equiv 0$.

[20]That is: all of R^3.

$$\text{so that, for } t = 0: \quad G_1 = \delta(x)\,\delta(y) \text{ and } (G_1)_t = 0, \tag{6.7}$$

$$G_2 = \frac{1}{2\pi}\left(\frac{H(t-r)}{\sqrt{t^2 - r^2}}\right),$$

$$\text{so that, for } t = 0: \quad G_2 = 0 \qquad \text{and } (G_2)_t = \delta(x)\,\delta(y), \tag{6.8}$$

where $r = \sqrt{x^2 + y^2}$, both G_1 and G_2 solve $u_{tt} = u_{xx} + u_{yy}$, and $H(\zeta) = \frac{1}{2}(1 + \text{sign}(\zeta))$ is the Heaviside step function.

HINT-3. Note that $\delta(z - \beta\,t) = \delta(\beta\,t - z) = \dfrac{1}{\sqrt{\beta^2 - 1}}\,\delta\left(\dfrac{\beta\,t - z}{\sqrt{\beta^2 - 1}}\right)$.

Remark 6.6 *You should note that, at the conic wave-front, the solution develops a very large amplitude (in fact: it is singular there). This is what triggers the blue "glow". In fact, the singularity is caused by the point source (i.e.: a delta forcing) approximation. For a small, but finite size, source the solution will develop a large amplitude at the wave front, but will not be singular.*
In the sonic boom case (supersonic propagation of a point source in, say, air) there is no infinite fields anywhere: at the wave front a shock wave appears[21] *(which cuts off the infinities). Furthermore: the shock wave is a "robust" object: it does not disappear when a finite size source is used.*

6.3 Statement: Moving point source in 1-D.

Situations where one has a moving source in the context of wave propagation are quite common. In particular — when the source is compact and one is only interested in the resulting wave pattern far away from the source[22] — one can often simplify the question by assuming a point source. Here we consider a very simple example of this type, in 1-D and for a scalar first order equation with constant coefficients (homogeneous media). We also assume "trivial" initial conditions.

When the equation is also linear, the problem is very simple, and the only (mildly) interesting effect that occurs is that of "resonance" when the source moves at the characteristic speed. The mathematical problem in this case is

$$u_t + c\,u_x = \delta(x - s\,t) \quad \text{and} \quad u(x,\,0) = 0, \tag{6.9}$$

[21] In some sense, the nonlinearity regularizes the solution: shocks do not involve infinite values of physical quantities.
[22] Distances much greater than the source size.

where c is the wave speed, s is the source speed (both constants), and $\delta(\cdot)$ is Dirac's delta function. **Show that (6.9) is equivalent to**

$$u_t = \delta(x - v\,t) \quad \text{and} \quad u(x,\,0) = 0, \tag{6.10}$$

for some constant v. Then **solve (6.10) for all possible values of v.** What happens at resonance?

The situation becomes much more interesting when the equation is nonlinear. Then the source can produce (or not) a precursor shock moving ahead of it — depending on the source speed and strength, and the (unrealistic) growth of the linear response in the resonant case is suppressed. As an example, consider the problem for the conserved density u

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \delta(x - c\,t) \quad \text{and} \quad u(x,\,0) = 0, \tag{6.11}$$

where c is a constant. **Solve this problem for all possible values of c.**

HINTS

H1. The solution responds to the delta-function forcing on the right with a discontinuity along $x = c\,t$. The discontinuity is such that the derivatives (interpreted in the weak sense) produce the delta function. Namely

$$-c\,[u] + \frac{1}{2}\,[u^2] = 1, \tag{6.12}$$

where $[\,]$ = jump across discontinuity (value ahead minus value behind). Specifically, if u_a is the value of u immediately ahead of the discontinuity, and u_b is the value immediately behind it, then: $[u] = u_a - u_b$ and $[u^2] = u_a^2 - u_b^2$.

Note that **not all the solutions to this equation are acceptable.** The next hint, and remark **6.7**, deal with this issue.

H2. Characteristics converge into shocks. However, the discontinuity along $x = c\,t$ is not a shock, but the response to a point forcing: the characteristics enter on one side of $x = c\,t$, and exit on the other. The only exception is when they enter/exit on one side and are parallel on the other — see remark **6.7**. But **the characteristics never converge on both sides of $x = c\,t$.**

H3. The characteristics for the un-forced equation are: $\dfrac{dx}{dt} = u$, along which $\dfrac{du}{dt} = 0$. Hence, *the initial value $u(x,\,0) = 0$ will persist at any given point x, till affected by something that makes the characteristic equations fail — namely: either a shock wave or the delta-function forcing.*

H4. The solution to (6.11) is rather simple. It is made up by constant strength/speed shocks, regions where u is constant, and rarefaction fans. Further, it is a function of x/t only — **why?**

H5. The shock conditions for (6.11) reduce to: (i) Shock speed is the average of u across the discontinuity. (ii) The value of u behind the shock is larger than the value ahead.

H6. The rarefaction fans for (6.11) are solutions where all the characteristics determining u emanate from a single point in space time and "fan" out.

Remark 6.7 *Here we elaborate on the subject matter of hints* **H1** *and* **H2**. *Specifically: what restrictions the solutions of the jump equation in (6.12) must satisfy — which is what* **H2** *is all about. Our objective is to understand the behavior of the characteristics for the equation in (6.11) at/near the location $x = c\,t$ of the delta function forcing. To do this, we consider (6.11) as the limit of*

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = f_\epsilon(x - c\,t) \quad \text{as} \quad \epsilon \to 0, \tag{6.13}$$

where $f_\epsilon(z)$ is a smooth, positive function, with total unit area, vanishing outside $|z| < \epsilon$. The characteristic equations for this problem are

$$\frac{dx}{dt} = u, \quad \text{along which} \quad \frac{du}{dt} = f_\epsilon(x - c\,t). \tag{6.14}$$

Then, as long as the characteristics are outside the forcing region $c\,t - \epsilon < x < c\,t + \epsilon$, they are straight lines — along which u is constant. When they enter the forcing region, on the other hand, they accelerate (as u increases). Hence, the following situations (and nothing else) can arise:

6.7-a. Characteristic enters the forcing region from the left, with $u > c$.

Then u starts increasing, the characteristic speeds up and leaves on the right side of the forcing region, carrying a larger value of u. The $\epsilon \to 0$ limit of this situation is (6.15) below.

6.7-b. Characteristic enters the forcing region from the left, with u barely above c; i.e.: $u = c + O(\epsilon)$.

Similar to item **6.7-a**, *but the $\epsilon \to 0$ limit is: Immediately behind $x = c\,t$, $u = c$ and the characteristics are parallel to the path of the delta function. Immediately ahead of $x = c\,t$, $u > c$ and the characteristics exit (to the right) from the path of the delta function. See (6.16) below.*

6.7-c. Characteristic is overtaken by the forcing region, and enters it from the right, with $u < c$ and sufficiently far below.

Then, once inside the forcing region, u starts increasing and the characteristic speeds up. However, before the value of u along the characteristic reaches c, the characteristic reaches the back of the forcing region, and exits it. The $\epsilon \to 0$ *limit of this situation is (6.17) below.*

6.7-d. Same as item **6.7**-c, but the value of u (when the characteristic enters the forcing region from the right) is just critical.

Then the characteristic just barely makes it out (from the back) of the forcing region. The $\epsilon \to 0$ *limit of this situation is (6.18) below.*

6.7-e. Same as item **6.7**-c, but the value of u (when the characteristic enters the forcing region from the right) is too close to c.

Then, once inside the forcing region, u starts increasing, the characteristic speeds up, and grows beyond c. Thus the characteristic will end up exiting the forcing region from the same side it entered. In the $\epsilon \to 0$ *limit of this, the characteristic "bounces back" (with a higher value of u) into the ahead of the path of the delta function. But this creates a multiple valued region for the solution ahead of the delta, which means that this is an inconsistent situation, and cannot occur.*[23]

Thus, in terms of u_a *and* u_b *(values immediately ahead — respectively behind – the discontinuity),* **these are the (only) acceptable possibilities for the solutions to equation (6.12):**

$$\text{Case 1:} \quad u_a > u_b > c. \tag{6.15}$$

$$\text{Case 2:} \quad u_a > u_b = c. \tag{6.16}$$

$$\text{Case 3:} \quad u_a < u_b < c. \tag{6.17}$$

$$\text{Case 4:} \quad u_a < u_b = c. \tag{6.18}$$

[23] If you try to solve equation (6.12) with a value of $u_a < c$ that is too close to c, you will see that then u_b would have to be complex, with a nonzero imaginary part.

6.4 Statement: Nonlinear diffusion from a point seed.

Solutions to linear equations with delta function data are particularly important because, via the superposition principle, they can be used to produce formulas for the solution with general data. In addition, because of the simplicity of the data, such solutions tend to have special symmetries that enormously reduce the complexity of the problem to be solved — e.g.: see sub-subsection 6.4.1.
For nonlinear situations, because of the lack of a superposition principle, the solutions to problems with delta function data are not as fundamental. Nevertheless, the fact is that delta data introduces (just as in linear case) symmetries that sometimes make such problems solvable. Hence, since very few exact solutions are available for nonlinear equations, these solutions can be very interesting, and a useful learning tool. In addition, they often describe interesting asymptotic limits — such as (for example) the field distribution due to a localized source, far away from the source.
In this exercise we consider the exact solutions that can be obtained for (certain kinds of) nonlinear diffusion set-ups, when the initial data corresponds to a point concentration of diffusing "stuff" (a "point seed"). In appropriate non-dimensional units, the problem is:

$$u_t = \mathrm{div}\left(|u|^n \nabla u\right) \quad \text{for } \ t > 0, \ \ \vec{x} \in R^d, \quad \text{and } \ u(\vec{x},\,0) = \delta(\vec{x}), \tag{6.19}$$

where ∇ is the gradient operator in d-dimensions, and $n \geq 0$ is a constant.

Part 1. Solve the problem in (6.19) for $n = d = 2$ — see hint **6.1** and sub-subsection 6.4.1.

Part 2. Solve the problem in (6.19) for $n = 1$ and $d = 2$ — see hint **6.2**.

Part 3. Solve the problem in (6.19) for $n > 0$ arbitrary and any $d = 1,\, 2, \ldots$

Part 4. Does (6.19) make sense if $n < 0$? Watch out, this is trickier than it looks!

Hint 6.1 *In linear diffusion problems there is no upper bound to the velocity at which things can diffuse — though the amount of "stuff" diffusing at high velocity is very, very, small. This gives rise to the phenomena seen in sub-subsection 6.4.1 for the solution of (6.19) in the linear case $n = 0$. Namely: the* solution is non-zero everywhere for any $t > 0$ *— albeit very small for large $\vec{x}$.*
On the other hand, when $n > 0$, the diffusion coefficient $|u|^n$ vanishes with u. This allows for sharp fronts in the solutions, with $u = 0$ "ahead" of the front, and $u > 0$ behind it. Furthermore, the

solution is not even smooth at the front! For example:

$$u = \left\{ \begin{array}{ll} 0 & \textit{for} \quad x \geq s\,t \\ \sqrt{2\,s\,(s\,t - x)} & \textit{for} \quad x \leq s\,t \end{array} \right\} \quad \textit{solves} \quad u_t = \Big(u^2\,u_x\Big)_x\,, \tag{6.20}$$

where $s > 0$ is some constant. Notice that, while u_x for the solution above in (6.20) is singular along $x = s\,t$ — not even continuous, $u^2\,u_x$ is continuous, and gives a well defined[24] $(u^2\,u_x)_x$. *Hence, in order to make sense of solutions with sharp fronts, it is important that the term* $\mathrm{div}\Big(|u|^n \nabla u\Big)$ *in the equation be left as written, <u>without</u> being expanded.*

A "physical" explanation *of why the fronts are not only sharp, but also singular, is as follows: Start the solution so that there is some boundary separating a region with non-zero concentration ($u > 0$) from one with zero concentration. Because the diffusivity is very small when u is small (limiting zero for $u = 0$), if the concentration is smooth at the edge of the non-zero zone, then there is no diffusion across the edge, and the edge does not move. But stuff starts diffusing from inside towards the edge, making the concentration profile steeper and steeper at the edge. Eventually (when the slope becomes infinity at the edge) the stuff "slightly" behind the edge has a large enough concentration (hence diffusivity) to diffuse right across the edge, which can start moving — but only as long as it stays very steep (singular).*

You should expect the solutions to the problem in (6.19) to exhibit behaviors similar to the one above in (6.20). *The solutions will exhibit a sharp, expanding, boundary — with $u > 0$ inside and $u = 0$ outside. Further: u will have infinite steepness at this boundary.* Unlike linear diffusion, where the edge of a diffusing blotch of stuff is blurry, when the diffusion coefficient is nonlinear, the edge can be very sharp.

Hint 6.2 *The solutions to the problem in (6.19) all have $u \geq 0$ everywhere. Hence $|u|^n = u^n$.*

Hint 6.3 *When considering the answer to* **part 4**, *look at the example where $d = 1$ and $n = -1$, and examine the similarity solutions of the type used in parts 1 – 3. Do they provide satisfactory solutions? Can you generalize your findings for other values of d and n?*

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[24] Albeit singular along $x = s\,t$.

6.4.1 Example: Green function for the heat equation in R^d.

Consider the initial value problem for the diffusion equation in d dimensions, with initial conditions corresponding to a point concentration of diffusing "stuff" (a "point seed"). In appropriate non-dimensional units, the problem is:

$$u_t = \Delta u \quad \text{for } \ t > 0, \ \ \vec{x} \in R^d, \quad \text{and } \ u(\vec{x},\, 0) = \delta(\vec{x}), \tag{6.21}$$

where Δ is the Laplacian operator in d-dimensions.

The problem in (6.21) is clearly invariant under rotations. In addition, since $\delta(c\,\vec{x}) = c^{-d}\,\delta(\vec{x})$ for any constant $c > 0$, (6.21) is also invariant under the transformation $u(t,\, \vec{x}) \to c^d\, u(c^2\, t,\, c\,\vec{x})$. Hence, we expect the solution to the problem to satisfy the symmetry $c^d\, u(c^2\, t,\, c\,\vec{x}) = u(t,\, \vec{x})$. It follows then that

$$u = t^{-d/2}\, U(z) \quad \text{for some function U,} \quad \text{where } \ z = \frac{r}{\sqrt{t}}, \tag{6.22}$$

and r is the radial coordinate in R^d. Substitution of this into (6.21) yields

$$-\frac{1}{2}\, z^{1-d}\, \left(z^d\, U\right)' = -\frac{1}{2}\, (d\, U + z\, U') = z^{1-d}\, \left(z^{d-1}\, U'\right)', \tag{6.23}$$

where primes indicate derivatives with respect to z, and the following conditions should apply:

A. Odd derivatives of U vanish at the origin.

B. U vanishes as $z \to \infty$.

C. $1 = \displaystyle\int u\, dx_1\, dx_2\, \ldots\, dx_d = A_{d-1} \int_0^\infty U(z)\, z^{d-1}\, dz$, where A_{d-1} = area of unit sphere in R^d.

Condition **A** follows because the solution u must be smooth at the origin, while conditions **B** and **C** guarantee that $u \to \delta(\vec{x})$ as $t \downarrow 0$.

Equation (6.23) can be integrated once, to get $U' = -\dfrac{1}{2}\, z\, U$. Thus $U = c\, e^{-z^2/4}$, where the constant c follows from the integral condition in item **C** above. Hence

$$G_d = \frac{t^{-d/2}}{A_{d-1}\, \int_0^\infty e^{-z^2/4}\, z^{d-1}\, dz}\, e^{-\frac{r^2}{4\,t}} = \frac{t^{-d/2}}{2^{d-1}\, A_{d-1}\, \Gamma(\frac{d}{2})}\, e^{-\frac{r^2}{4\,t}}, \tag{6.24}$$

where Γ is the Gamma function, and G_d is the solution of (6.21). In particular:

- $A_0 = 2$ and $\Gamma(1/2) = \sqrt{\pi}$. Thus .... $G_1 = \dfrac{1}{\sqrt{4\,\pi\,t}}\, e^{-x^2/(4\,t)}$.

- $A_1 = 2\,\pi$ and $\Gamma(1) = 1$. Thus .... $G_2 = \dfrac{1}{4\,\pi\,t}\,e^{-r^2/(4\,t)}$.
- $A_2 = 4\,\pi$ and $\Gamma(3/2) = \dfrac{1}{2}\sqrt{\pi}$. Thus .... $G_3 = \dfrac{1}{(4\,\pi\,t)^{3/2}}\,e^{-r^2/(4\,t)}$.

In fact, it is easy to see (directly) that it should be

$$G_d = G_1(x_1,\,t)\,G_1(x_2,\,t)\,\ldots\,G_1(x_d,\,t) = \frac{1}{(4\,\pi\,t)^{d/2}}\,e^{-r^2/(4\,t)}. \tag{6.25}$$

This follows because $\delta(\vec{x}) = \delta(x_1)\,\delta(x_2)\,\ldots\delta(x_d)$, and the expression in (6.25) satisfies the diffusion equation in d-dimensions. Comparing (6.25) with (6.24) then yields

$$A_{d-1} = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}. \tag{6.26}$$

This provides a **rather indirect way to compute the area of the sphere in d-dimensions!**

Remark 6.8 General initial value problem for the linear diffusion equation in (6.21). *Using the solution G_d above in (6.25), we can write*

$$u = \int_{R^d} G_d(\vec{x}-\vec{s},\,t)\,u(\vec{s},\,0)\,ds_1\,ds_2\,\ldots ds_d, \tag{6.27}$$

which provides an explicit formula for the solution of the initial value problem for the heat equation in R^d. For solutions that decay at infinity (including derivatives) fast enough, a proof of **uniqueness** *is rather simple. Using integration by parts, one can see that any such solution will also satisfy*

$$\frac{d}{dt}\,\frac{1}{2}\int_{R^d} u^2 dx_1\,\ldots\,dx_d = -\frac{1}{2}\int_{R^d}(\nabla u)^2 dx_1\,\ldots\,dx_d < 0.$$

Hence, if u vanishes at time $t = 0$, it will vanish for all times. Thus, any two solutions with the same initial values must be equal, as their difference satisfies the equation with zero initial data.

6.4.2 Moisture transport in porous media.

Moisture transport in an unsaturated porous media[25] provides an example of a nonlinear diffusion equation of the type studied in this exercise. In many cases, the underlying flow can be modeled by a nonlinear diffusion equation of the form $u_t = \mathrm{div}\,(\mu(u)\,\nabla u)$, where $u \geq 0$ is the water volume

[25] Multi-phase flow through soil, rock, filters, wood, concrete and many other natural and man-made materials.

fraction or saturation, while $\mu(u)$ is the moisture diffusivity — a property of the porous medium. For many applications, the diffusivity has been correlated experimentally with a power law of the form $\mu(u) \propto u^n$. A few relevant references are:

M. A. Heaslet and A. Alksne.
Diffusion from a fixed surface with a concentration-dependent coefficient.
J. Soc. Indust. Appl. Math., 9(4):584–596, 1961.

R. E. Pattle.
Diffusion from an instantaneous point source with a concentration-dependent coefficient.
Quart. J. Mech. Appl. Math., 12(4):407–409, 1959.

L. F. Shampine.
Concentration-dependent diffusion. II. Singular problems.
Quart. Appl. Math., pp. 287–293, Oct. 1973.

7 Shock Jump and Entropy Conditions.

7.1 Statement: Lax entropy condition for scalar convex conservation laws, and information loss.

Introduction.

Consider the question of measuring the amount of variability of an arbitrary function $f = f(x)$ — its "wavy-ness", as it were. By this we mean that we want to define an operation $\mathcal{I}$ on functions such that: $\mathcal{I}(f;\, a,\, b)$ is a number that provides a "measure" of the amount of "wavy-ness" contained by the function f in the interval $a < x < b$. The following seem to be desirable properties that we would like $\mathcal{I}$ to have:

P1. $\mathcal{I}(f;\, a,\, c) = \mathcal{I}(f;\, a,\, b) + \mathcal{I}(f;\, b,\, c)$ for $a < b < c$.

P2. $\mathcal{I}$ should grow as f becomes more wavy, and be at its minimum when f is a constant. In particular, if $f = \alpha + s\, g(x)$, where α is a (fixed) constant, s is a parameter, and g is a function with vanishing mean[26] over $a < x < b$, then $\mathcal{I}(f;\, a,\, b)$ should be a *strictly increasing* function of $|s|$, with a *minimum* at $s = 0$.

[26]In other words: g encodes the "shape" of the graph of f, s its amplitude, and α its mean.

P3. It would also be nice if we could require that $\mathcal{I}(f;\, a,\, b) = \mathcal{I}(f+\mu;\, a,\, b)$ for any constant μ, since adding a constant to f should not change its wavy-ness. However, as we will see below, we will have to give up on this property.

Unfortunately, these are not properties that characterize $\mathcal{I}$ uniquely, so there are many possible choices for what $\mathcal{I}$ should be — not surprising, since "wavy-ness" is a rather ambiguous concept.

1st. From item **P1** it seems reasonable to conclude that we *should take* $\mathcal{I}$ *as an integral of some local density* Ψ — with Ψ depending only on the local values of f — in other words, Ψ *is a function of* f *and its derivatives.*

2nd. In order to have $\mathcal{I}$ defined even for functions f that are not smooth, we eliminate any dependence of Ψ on derivatives, and take $\Psi = \Psi(f)$. Thus

$$\mathcal{I}(f,\, a,\, b) = \int_a^b \Psi(f(x))\, dx, \quad \text{where } \Psi \text{ is some fixed function.} \tag{7.1}$$

Then (assuming that Ψ is smooth), in order to enforce **P2**, we note that $\mathcal{I} = \displaystyle\int_a^b \Psi(\alpha + s\, g(x))\, dx$ yields $\left.\dfrac{d\mathcal{I}}{ds}\right|_{s=0} = 0$ and $\dfrac{d^2\mathcal{I}}{ds^2} = \displaystyle\int_a^b g^2(x)\, \Psi''(\alpha + s\, g(x))\, dx$. Since g is arbitrary, it follows that

$$\textbf{P2} \text{ applies if } \Psi \text{ is convex: } \quad \frac{d^2\, \Psi}{df^2} = \Psi'' \geq C_\Psi > 0, \tag{7.2}$$

where C_Ψ is some constant.

Remark 7.1 *It should be clear that $\mathcal{I}$, as defined above by (7.1) and (7.2),* does not *satisfy* **P3***. If, on the other hand, we allow the density Ψ to depend on derivatives of f, then* **P3** *can be satisfied. For example*

$$\mathcal{I}(f,\, a,\, b) = \int_a^b |f'(x)|\, dx, \tag{7.3}$$

is called the variation *of f, and satisfies all three properties* **P1** – **P3**. *This definition can be extended to functions f without a derivative, but then life gets complicated — so* we will avoid this here.

Remark 7.2 *Possible "interpretations" of the meaning of $\mathcal{I}$:*

I1. *In some (very rough) sense, the amount of* information *encoded in some function $f = f(x)$ is determined by how "wavy" the function is. A constant encodes just one number: its value. A sinusoidal can be characterized by an amplitude, a wavelength, and a phase. And so on: the more complicated the wave shape, the more parameters (information) that are needed to fully describe it. On the other hand, one could argue that any two sinusoidals have the same amount of information — independent of, say, their amplitude. However, "wavy-ness" increases as the amplitude grows — see item* **P2**.

I2. *Another interpretation for $\mathcal{I}$ is that it is some kind of* energy*, as follows: Imagine that the function f describes the density of something that "likes" to be at some given equilibrium state $f \equiv f_e =$ constant, and generates forces when away from equilibrium. Then the "wavy-ness", as defined by (7.1) is a measure of the energy needed to maintain a density f, with Ψ the potential energy of the forces generated.*

The problem.

Consider a scalar conservation law (with shocks), of the form

$$\rho_t + q_x = 0, \quad \text{with} \quad q = q(\rho) \text{ smooth and convex: } \frac{d^2q}{d\rho^2} \geq C_q > 0, \tag{7.4}$$

where C_q is some constant. Then **SHOW THAT:**

The Lax entropy condition on shocks is equivalent to the statement that $\mathcal{I}$, as defined by (7.1) and (7.2), is decreasing across shocks. (7.5)

In other words, you have to show that $\mathcal{I}$ is decreased by the presence of a shock if and only the jump across the shock in ρ is decreasing — since, for q convex, this corresponds to the characteristics converging into the shock path.

In order to do this problem, proceed as follows;

1. **Show** that, if $\rho = \rho(x, t)$ is a **smooth solution** of (7.4), then $\Psi(\rho)$ is "conserved". Namely:

$$\Psi_t + h_x = 0, \quad \text{with} \quad h = h(\rho) \text{ defined by } \frac{dh}{d\rho} = c(\rho)\,\frac{d\Psi}{d\rho}, \tag{7.6}$$

where $c = \dfrac{dq}{d\rho}$ is the characteristic speed for the conservation law. Thus

$$\frac{d}{dt}\mathcal{I}(\rho, a, b) = h_a - h_b \quad \text{if } \rho \text{ is smooth for } a \leq x \leq b, \tag{7.7}$$

where $h_a = h(\rho(a,\, t))$ and $h_b = h(\rho(b,\, t))$. This identifies h as the "flux" for Ψ.

2. Consider now the situation where the solution $\rho = \rho(x,\, t)$ is smooth for $a \le x \le b$, **except for** a simple jump discontinuity at $a < x = \sigma(t) < b$, satisfying the Rankine-Hugoniot jump condition[27]

$$\dot{\sigma} = \frac{d\sigma}{dt} = \frac{[q]}{[\rho]} = \frac{q_R - q_L}{\rho_R - \rho_L}, \quad \text{with} \quad \rho_R \neq \rho_L. \tag{7.8}$$

Here the subscript R indicates evaluation on the right side of the discontinuity — i.e. at $x = \sigma + dx$, and the subscript L indicates evaluation on the left side of the discontinuity — i.e. at $x = \sigma - dx$. Then **show that (7.7) has to be modified as follows**

$$\frac{d}{dt}\mathcal{I}(\rho,\, a,\, b) = h_a - h_b + \underbrace{h_R - h_L - \dot{\sigma}\,(\Psi_R - \Psi_L)}_{D}, \tag{7.9}$$

where D is the contribution from the discontinuity to the rate of change of $\mathcal{I}$. Hence **(7.5) is equivalent to**

$$\operatorname{sign}(D) = \operatorname{sign}(\rho_R - \rho_L). \tag{7.10}$$

3. Show that (7.10) holds.

Hint: Note that, using (7.8), we can write

$$D = \frac{1}{\rho_R - \rho_L}\,\Big((h_R - h_L)(\rho_R - \rho_L) - (q_R - q_L)(\Psi_R - \Psi_L)\Big) = \frac{M}{\rho_R - \rho_L}, \tag{7.11}$$

where M is defined by the formula. Thus, to prove (7.10), you have to show that

$$M > 0 \quad \text{for} \quad \rho_R \neq \rho_L. \tag{7.12}$$

This you can do by keeping ρ_L fixed (but arbitrary) and considering M as a function of ρ_R. Then (7.12) follows because

$$\left.\begin{array}{lll} \textbf{3a.} & \dfrac{d^2M}{d\rho_R^2} = \dfrac{dM}{d\rho_R} = M = 0 & \text{for } \ \rho_R = \rho_L. \\[2ex] \textbf{3b.} & \dfrac{d^2M}{d\rho_R^2} > 0 & \text{for } \ \rho_R \neq \rho_L. \end{array}\right\} \tag{7.13}$$

It is in **3b** where the convexity of both q and Ψ plays its role. In particular, notice that:

For a convex function, the tangent line through any point in its graph is strictly below the graph of the function away from the tangent point. (7.14)

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[27] These guarantee that ρ is conserved.

7.2 Statement: Zero viscosity limit in scalar convex conservation laws and dissipation.

Consider a scalar convex conservation law, with a small amount of "viscosity" added. Namely:

$$\rho_t + q_x = \nu\,\rho_{x\,x}, \quad \text{with} \;\; q = q(\rho) \;\; \text{smooth and convex:} \;\; \frac{d^2 q}{d\rho^2} \geq C_q > 0, \;\; \text{and} \;\; \rho(x,\,0) = f(x), \tag{7.15}$$

where C_q is some constant, $0 < \nu \ll 1$, f is smooth, and f and all its derivatives vanish (rapidly) as $|x| \to \infty$. We want to investigate the behavior, as $\nu \to 0$, of the solution to this problem. In particular, **we want to compute**

$$\mathcal{I} = \mathcal{I}(t) = \int_{-\infty}^{\infty} \Psi(\rho(x,\,t))\,dx, \quad \text{as} \;\; \nu \to 0, \tag{7.16}$$

where Ψ is a smooth convex function[28] such that $\Psi(0) = 0$. Furthermore, let $h = h(\rho)$ be defined by

$$\frac{dh}{d\rho} = c(\rho)\,\frac{d\Psi}{d\rho} \;\; \text{and} \;\; h(0) = 0, \quad \text{where} \quad c = \frac{dq}{d\rho}. \tag{7.17}$$

Note 1. We will assume that the solution $\rho = \rho(x,\,t)$ to (7.15) exists, that it is smooth, that ρ and all its derivatives vanish (rapidly) as $|x| \to \infty$, and that ρ is unique (within this class).

Note 2. Here we will use concepts introduced in the problem Lax Entropy condition for scalar convex conservation laws and information loss — you should read the statement for this exercise. This will also serve the purpose of giving meaning to $\mathcal{I}$ as a measure of the "wavy-ness" of the solution.

Multiplying by $\dfrac{d\Psi}{d\rho}$ the equation $\rho_t + c(\rho)\,\rho_x = \nu\,\rho_{x\,x}$ satisfied by ρ, and using (7.17), we obtain

$$\Psi(\rho)_t + h(\rho)_x = \nu\,\Psi'(\rho)\,\rho_{x\,x}, \tag{7.18}$$

where the prime denotes differentiation with respect to ρ. Integrating this equation then yields

$$\frac{d\mathcal{I}}{dt} = \nu \int_{-\infty}^{\infty} \Psi'(\rho)\,\rho_{x\,x}\,dx = -\nu \int_{-\infty}^{\infty} \Psi''(\rho)\,\rho_x^2\,dx < 0, \tag{7.19}$$

which shows that $\mathcal{I}$ is, always, a decreasing function of time.

From (7.19) is seems natural to conclude that: in the limit $\nu \to 0$, $\mathcal{I}$ becomes constant. However, **this is false!** As $\nu \to 0$, the solution to (7.15) develops thin transition layers (shocks) where the

[28] Hence $\dfrac{d^2\,\Psi}{d\rho^2} \geq C_\Psi > 0$, for some constant C_Ψ.

derivatives become large, so that the right hand side in (7.19) does not vanish as $\nu \to 0$. **Your task here is to check this fact.** *Proceed as follows.*

First, consider the case where the solution to (7.15) develops a single shock as $\nu \to 0$, along some path $x = \sigma(t)$. Then,

$$\left.\begin{array}{l} \text{For } 0 < \nu \ll 1 \text{, the solution near } x = \sigma \text{ (for any fixed time } t\text{) can be described} \\ \text{by a traveling wave solution to the equation in (7.15), of the form} \\ \qquad \rho \sim \mathcal{R}(z), \quad \text{where} \quad z = \dfrac{x - s\,t}{\nu} \quad \text{and} \quad s = \dfrac{d\,\sigma}{d\,t}, \\ \text{with limits } \rho_R \text{ as } z \to \infty \text{ and } \rho_L \text{ as } z \to -\infty \text{. On the other hand, away from} \\ x = \sigma \text{, the derivatives of } \rho \text{ remain bounded as } \nu \to 0. \end{array}\right\} \tag{7.20}$$

Use (7.20) and (7.19) to **compute the limit,** as $\nu \to 0$, of $\dfrac{d\mathcal{I}}{dt}$. You should be able to **show that**

$$\frac{d\mathcal{I}}{dt} \quad \longrightarrow \quad h_R - h_L - s\,(\Psi_R - \Psi_L), \tag{7.21}$$

where the subscript R indicates evaluation at ρ_R, and the subscript L indicates evaluation at ρ_L. *Notice that this is the same formula for the contribution to the rate of change of $\mathcal{I}$ by a shock* listed in the statement to the problem Lax Entropy condition for scalar convex conservation laws and information loss — see equation (7.9) there.

Hint. Consider the first integral in (7.19). Away from $x = \sigma$, the derivatives of the solution remain bounded, and the contribution of these regions to the integral vanishes as $\nu \to 0$. Hence we can limit the integration to a small region near the shock, namely

$$\frac{d\mathcal{I}}{dt} \approx \nu \int_{\sigma-\epsilon}^{\sigma+\epsilon} \Psi'(\rho)\,\rho_{x\,x}\,dx, \quad 0 < \nu \ll \epsilon \ll 1, \tag{7.22}$$

where the traveling wave approximation $\rho \sim \mathcal{R}(z)$ in (7.20) can be used. From this, using the o.d.e. that $\mathcal{R}$ satisfies,[29] the result in (7.21) follows.

Second, if the solution to (7.15) develops more than one shock as $\nu \to 0$, then a calculation as the one above can be done near each of the shock locations $x = \sigma_\ell(t)$, obtaining as the result that the right hand side in (7.21) is replaced by a sum including the contributions for each one of the shocks.

[29] Note that **you do not need to have an explicit formula for** $\mathcal{R}$. Knowing the o.d.e. that $\mathcal{R}$ satisfies, and knowing the limits as $z \to \pm\infty$ for $\mathcal{R}(z)$, **should be enough.**

Remark 7.3 *The result in this problem illustrates a persistent phenomena in nonlinear p.d.e. that incorporate "small" perturbations involving the highest derivatives in the equation. Then, as the perturbations vanish, their effects on the solution* **do not.** *For example: in high Reynolds number flows thin boundary layers can form (mostly near walls), where viscous effects dominate, and contribute a finite amount (that does not go away as the Reynolds number goes to infinity) to the flow behavior. In other cases high frequency (thin) structures appear over large regions of the solution, which again do not go away as the perturbations vanish. This second type of situation is quite often associated with open problems where a good mathematical theory is lacking — e.g.: collision-less shocks in plasmas, turbulence, etc.*

7.3 Statement: Entropy conditions for scalar (non convex) problems.

Consider the following conservation law, for the single scalar function $\mathrm{v} = \mathrm{v}(x,\, t)$,

$$\mathrm{v}_t + p_x = 0, \quad \text{where} \quad p = \mathrm{v}\,(\mathrm{v}^2 - 1), \tag{7.23}$$

v is the density for some conserved quantity, and p is the flux. Shocks for this equation, if allowed, should satisfy the Rankine-Hugoniot jump conditions

$$-s\,[\mathrm{v}] + [p] = 0, \quad \Longleftrightarrow \quad s \;=\; \frac{[p]}{[\mathrm{v}]} \;=\; \mathrm{v}_a^2 + \mathrm{v}_a\,\mathrm{v}_b + \mathrm{v}_b^2 - 1, \tag{7.24}$$

where $[\cdot]$ denotes the jump across the discontinuity of the enclosed quantity, v_a (resp. v_b) is the value of v immediately ahead of (resp. behind) the discontinuity, and $\mathrm{v}_a \neq \mathrm{v}_b$.

The objective of this exercise is to find what additional restrictions ("entropy" conditions) the solutions to (7.24) must satisfy to produce acceptable as shocks. We will try two approaches. **This problem has 3 "tasks", do them all!**

"STABILITY" ANALYSIS: Let v be a steady state "solution" to (7.23), with a shock satisfying (7.24). Namely, let

$$\mathrm{v}(x,\, t) = a \;\text{ for }\; x \geq st, \quad \text{and} \quad \mathrm{v}(x,\, t) = b \;\text{ for }\; x \leq st, \tag{7.25}$$

where $a \neq b$ are constants and $s = a^2 + a\,b + b^2 - 1$. Then:

$$\left.\begin{array}{r}\textit{Declare (7.25) an "acceptable" solution if and only if the linear problem that}\\ \textit{results when infinitesimal perturbations to (7.25) are considered is well posed.}\end{array}\right\} \tag{7.26}$$

In other words: consider solutions to (7.23 – 7.24) of the following form:

$$\mathrm{v}(x,\,t) = a + A(x,\,t) \;\text{ for }\; x \geq st + r(t), \quad \text{and} \quad \mathrm{v}(x,\,t) = b + B(x,\,t) \;\text{ for }\; x \leq st + r(t), \tag{7.27}$$

where A, B, and r are infinitesimal. This will lead to a *system of linear equations for A, B, and r — with initial conditions*

$$\left.\begin{array}{lcll} A(x,0) & = & A_0(x) & \text{defined for } x \geq 0 \text{ only,} \\ B(x,0) & = & B_0(x) & \text{defined for } x \leq 0 \text{ only,} \\ r(0) & = & r_0. & \end{array}\right\} \tag{7.28}$$

Then the solution in (7.25) is accepted if and only if the initial value problem for A, B, and r is well posed (solutions exist and are unique).

Your task #1: Derive the equations satisfied by A, B, and r, and find which conditions must be imposed on a and b so that the initial value problem for A, B, and r is well posed.

HINT-1. Existence will be obvious.[30] However: be careful that you check uniqueness!

HINT-2. Interpret the conditions on a and b graphically. In the p-v plane,[31] consider the two curves: $p = \mathrm{v}\,(\mathrm{v}^2 - 1)$, and the secant line going through $(\mathrm{v} = a,\, p = p(a))$ and $(\mathrm{v} = b,\, p = p(b))$ — note that *the shock speed s is the slope of this secant line!* **What do the conditions on a and b mean in terms of this picture?**

ZERO VISCOSITY LIMIT: Consider equation (7.23) as the $\epsilon \downarrow 0$ of the equation

$$\mathrm{v}_t + p_x = \epsilon\,\mathrm{v}_{xx}. \tag{7.29}$$

Then the solution in (7.25) is accepted if and only there is a solution of this equation whose limit is (7.25). To be more precise: a solution to (7.29) of the form

$$\mathrm{v} = y\left(\frac{x - s\,t}{\epsilon}\right) \tag{7.30}$$

is sought such that $y(z) \to a$ as $z \to \infty$ and $y(z) \to b$ as $z \to -\infty$ (notice that y satisfies a second order O.D.E., which can be easily integrated once). If such a solution exists, then (7.25) is accepted.

Your task #2: Carry out the calculation above, and find out under what restrictions on a and b a solution such as in (7.30) can be found. Compare your answers with those from the task #1.

[30] The problem is very simple, and the solutions can be written explicitly.

[31] Let the horizontal axis be v, and the vertical be p.

HINT-3. You will find yourself having to inspect an O.D.E. of the form $y' = F(y)$. You do not need to be able to solve this equation explicitly to answer the question. Just notice that the solutions to $y' = F(y)$ connect consecutive zeros of F from $z = -\infty$ to $z = +\infty$, with the direction of the connection determined by the sign of F between the two zeros.[32] As in hint-2, a geometrical approach helps: think of F as the difference between $p(y) = y\,(y^2 - 1)$ and a straight line.

Your task #3: Imagine that, in equation (7.23), we replace $p = \mathrm{v}\,(\mathrm{v}^2 - 1)$ by $p = (\mathrm{v}^2 - 1)^2$. How will this change the answers? In particular — **note that this is the ONLY questions whose answer is required, but you must justify your answer:** *(i) Will the conditions resulting from a stability analysis, and those resulting from a zero viscosity limit, be the same? (ii) If not, which set of conditions is better?*

HINT-4. If you followed the advice of the prior hints, and interpreted your analysis for the prior two tasks geometrically, you should be able to perform this last task with nearly zero algebra. If not, you probably missed some key issue with tasks #1 or #2. Go back and take a second look. What your prior analysis should have shown you is that the *zero viscosity limit is a "global" criteria* — depending on the values of $p(\mathrm{v})$ for the whole range between a and b, while the *stability analysis is "local" criteria* — depending on the behavior of p near a and b only.

Remark 7.4 *Equation (7.23) is a not-too unrealistic (qualitative "toy" model) for some of the problems that arise when shock waves and phase transitions are (simultaneously) involved. Just to give you a little bit of the flavor of the connection: for a polytropic gas the equation of state $e = p\,\mathrm{v}/(\gamma - 1)$ gives isothermals and isentropes that are convex curves in the p-v plane.*[33] *However, when van der Waals forces are added, the curves become non-convex and develop local maximums and minimums like $p = p(\mathrm{v})$ above in (7.23), which lead to complicated restrictions on what solutions of the Rankine-Hugoniot conditions should be allowed, etc. Of course, (7.23) is far too simple to capture anymore than a few of the issues that arise in the "real" problem.*

[32] The nature of $\lim_{z \to \pm\infty} y$ depends on the zero: simple (higher order) zeros give exponential (algebraic) behavior.

[33] Actually, the important curves to consider are the Hugoniot curves, which are also convex.

7.4 Statement: Gas Dynamics strong shock conditions.

Consider the 1-D Euler equations of Gas Dynamics, and a right shock wave. In a frame of reference moving with the wave, the equations governing the propagation of the shock are

$$\begin{aligned}
\rho_0\, u_0 &= \rho_1\, u_1 \;<\; 0 && \text{(conservation of mass)}, && (7.31)\\
\rho_0\, u_0^2 + p_0 &= \rho_1\, u_1^2 + p_1 && \text{(conservation of momentum)}, && (7.32)\\
\rho_0\, u_0\, E_0 + p_0\, u_0 &= \rho_1\, u_1\, E_1 + p_1\, u_1 && \text{(conservation of energy)}, && (7.33)\\
\rho_0 &< \rho_1 && \text{(entropy)}, && (7.34)
\end{aligned}$$

where ρ is the gas density, u is the flow velocity, p is the pressure, $E = \dfrac{1}{2}\,u^2 + e$ is the energy per unit mass, e is the internal energy per unit mass, the subscript 0 (resp. 1) indicates values immediately ahead (resp. behind) the shock, and $\rho_0\, u_0 < 0$ because the flow is from right to left across the shock.

Equation (7.33) is equivalent to $\rho_0\, u_0 \left(\dfrac{1}{2}\,u_0^2 + h_0\right) = \rho_1\, u_1 \left(\dfrac{1}{2}\,u_1^2 + h_1\right)$ where $h = e + \dfrac{p}{\rho}$ is the enthalpy. Hence, using (7.31), we see that **equation (7.33) can be replaced by:**

$$\frac{1}{2}\,u_0^2 + h_0 \;=\; \frac{1}{2}\,u_1^2 + h_1. \tag{7.35}$$

Furthermore, assume a **strong shock in a polytropic gas**, so that

$$h = \frac{\gamma\, p}{(\gamma-1)\,\rho} \quad\Longleftrightarrow\quad a^2 = \gamma\,\frac{p}{\rho}, \qquad \text{and} \qquad M \;=\; -\,\frac{u_0}{a_0} \;\gg\; 1, \tag{7.36}$$

where $\gamma > 1$ is the ratio of specific heats, $a > 0$ is the sound speed, and M is the Mach number with respect to the flow ahead. Note that $M > 1$ is equivalent to equation (7.34).

DERIVE (approximate, leading order) expressions for u_1, ρ_1, and p_1 in terms of u_0, ρ_0, and p_0.

HINT: Use equation (7.36) to eliminate p and h — from equations (7.32) and (7.35) — in favor of a^2. Then use $M \gg 1$ to simplify the resulting equations. From then on, it should be smooth sailing.

PART II: rewrite the shock equations in terms of the coordinate frame where the fluid ahead of the shock is at rest, with $U > 0$ the shock velocity relative to the flow ahead.

7.5 Statement: Entropy conditions for the p-system.

Consider the following system of conservation laws (the p-System) in one space dimension

$$v_t - u_x = 0 \quad \text{and} \quad u_t + p_x = 0, \tag{7.37}$$

where $p = p(v)$ for $0 < v < \infty$. We are only interested in solutions where $v \geq 0$, and assume that

$$\left.\begin{array}{llllll} \textbf{(a)} & p & > & 0, & & \\ \textbf{(b)} & \dfrac{dp}{dv} & < & 0, & & \\ \textbf{(c)} & \dfrac{d^2p}{dv^2} & > & 0, & & \\ \textbf{(d)} & p & \to & 0 & \text{as} & v \to \infty, \\ \textbf{(e)} & p & \to & \infty & \text{as} & v \to 0. \end{array}\right\} \tag{7.38}$$

For example $p = v^{-\gamma}$, with $\gamma > 2$, has these properties. **An important conclusion** (needed for the answer) is that p **is strictly decreasing function of** v**, with a convex graph** $\Longrightarrow$ *any straight line tangent to the graph of* p *lies strictly below the graph,* except for the point of contact. The system in (7.37) is strictly hyperbolic, with characteristic speeds $\pm a(v)$, where $a = \sqrt{-dp/dv} > 0$.

Remark 7.5 (Examples of physical contexts where (7.37) applies).
As we saw in the lectures, the system in (7.37) can be used to model gas dynamics in one dimension (inside a narrow tube, say), when transport effects (viscosity and heat conductivity) can be ignored, and the motion is assumed isentropic (entropy identically constant). In this case v *is the specific volume,* u *is the flow speed,* p *is the pressure,* x *is the mass-Lagrangian coordinate, and* a *is the sound "speed" in the Lagrangian coordinates. In this case* x *has dimensions mass/area, and* a *has dimensions mass/(area* $\times$ *time) — where the area in question is the cross-sectional area of the tube where the motion occurs.*
The system in (7.37) is also a model for long waves in shallow channel with a flat horizontal bottom, where $v = 1/h$*,* h *is the water depth,* $p = \dfrac{g}{2}v^{-2}$ *is the integral over the depth of the hydrostatic pressure (divided by the water density),* g *is the acceleration of gravity,* u *is the depth averaged flow velocity, and* x *a Lagrangian coordinate measuring the volume (per unit width of the channel) of water from some fixed parcel of liquid — hence* x *has dimensions of area and* a *(the wave speed in Lagrangian coordinates) has dimensions of area over time.*

The system in (7.37) is also a model for the longitudinal vibrations of an homogeneous elastic rod. In this case x is a Lagrangian coordinate attached to each point along the rod, equal to the position of the point when the rod is at rest. Then, if d is the current position of each point, $v = d_x - 1$ is the strain, $u = d_t$ is the velocity, and $-\rho\, p = F(v)$ is the elastic restoring force — with ρ the density (mass per unit length) of the rod.

For the equations in (7.37), consider now a shock wave moving with constant speed $D > 0$ into a rest state ahead of the shock, where $0 < v = v_0 < \infty$ and $u = 0$. Let then $v = v_1$ and $u = u_1$ be the state behind the shock (this is the region $x < D\,t$ if the shock is at $x = 0$ for $t = 0$). The Rankine-Hugoniot jump conditions must apply across the shock, so that

$$D\,(v_1 - v_0) + u_1 = 0 \quad \text{and} \quad -D\,u_1 + (p_1 - p_0) = 0, \tag{7.39}$$

where $p_j = p(v_j)$. We now *take the position that we know the state ahead of the shock, and want to predict the state behind as a function of the shock speed D, for whatever speeds are allowed.*[34] Thus we re-write these equations in the equivalent form:

$$u_1 = D\,(v_0 - v_1) \quad \text{and} \quad D^2 = -\frac{p_1 - p_0}{v_1 - v_0}, \tag{7.40}$$

where the second equation gives v_1 implicitly as a function of D and v_0, and the first can be used to recover u_1. We will now investigate these equations. **SHOW THAT:**

1. Let $F = F(v_1)$ be the right hand side of the second equation in (7.40) — where v_0 is given and fixed. F is defined for $0 < v_1 < \infty$ and

   1a. $F \to \infty$ as $v_1 \to 0$. **1c.** $dF/dv_1 < 0$ for $0 < v_1 < \infty$.

   1b. $F \to 0$ as $v_1 \to \infty$. **1d.** $F(v_1) = a^2(v_0)$.

   Hint: To show **1.c**, use that the graph of p is convex — see the paragraph below (7.38).

2. From item **1**, we conclude that (this is obvious, you do not have to show it):

   2a. For every value $0 < D < \infty$ there is exactly one value $\infty > v_1 > 0$ satisfying (7.40). Furthermore: v_1 is a decreasing function of D.

   2b. If $D > a_0 = a(v_0)$, then $v_1 < v_0$ (and $u_1 > 0$) — the wave *compresses.*

   2c. If $D < a_0 = a(v_0)$, then $v_1 > v_0$ (and $u_1 < 0$) — the wave *expands.* Below, in item **4**, you are asked to show that *waves like this (expansion shocks) are not allowed.*

[34] Part of this problem will be to show that there is a minimum possible shock speed.

3. Notice that (7.40) implies that $-D^2$ is the slope of the secant line through the points $(v_0,\, p_0)$ and $(v_1,\, p_1)$ in the graph of p. From the convexity of this graph, conclude that:

3a. $v_1 < v_0 \quad\Longrightarrow\quad -a_1^2 < -D^2 < -a_0^2 \quad\Longrightarrow\quad a_1 > D > a_0$.

3b. $v_1 > v_0 \quad\Longrightarrow\quad -a_1^2 > -D^2 > -a_0^2 \quad\Longrightarrow\quad a_1 < D < a_0$.

4. From items **2** and **3** it follows that for a compressive right moving shock the right characteristics $\dfrac{dx}{dt} = a$ converge into the shock, while for an expansion shock they originate at the shock. Argue then that **expansion shocks violate causality, and therefore they are not acceptable solutions to the Rankine-Hugoniot jump conditions.** Only the compression shock solutions should be accepted.

Hint: the solution at any point in space-time is determined by the information carried by the two characteristics (one left and one right) arriving at that point. If only compression shocks are allowed, then these two characteristics will always connect the point to either initial data or boundary data. What happens if expansion shocks are allowed?

Remark 7.6 *Obviously, similar arguments apply to left moving shocks, where $D < 0$. In this case, again, only the compression branch solutions to the Rankine-Hugoniot equations are acceptable. These have the property that the left moving characteristics $\dfrac{dx}{dt} = -a$ converge into the shock: $-a_0 > D > -a_1$, where the subscript 0 refers to the state upstream of the shock (the left in this case), and the subscript 1 refers to the state downstream of the shock (the right in this case).*
The condition that the characteristics corresponding to the shock (right characteristics for a right shock, and left characteristics for a left shock) should converge into the shock is called a **Lax Entropy Condition.** *In the 1950's Lax introduced conditions of this type as conditions that shocks for general systems must satisfy so that causality is not violated. In the case of Gas Dynamics the Lax condition is equivalent to the statement that the entropy contents of a fluid parcel increases as it goes through the shock, hence the name.*

7.6 Statement: Shallow water - Energy dissipation at shocks.

Consider the *Shallow Water Wave* equations in 1-D over a flat horizontal bottom:

$$h_t + (h\,u)_x = 0 \quad \text{and} \quad (h\,u)_t + \left(h\,u^2 + \frac{1}{2}\,g\,h^2\right)_x = 0, \tag{7.41}$$

where the first equation expresses the conservation of mass (or volume, since the density is constant), the second expresses the conservation of momentum, and g is the acceleration of gravity.

Part I

For solutions without *hydraulic jumps* (i.e.: shocks), **show that** the *mechanical energy* is also conserved, and derive an equation of the form

$$(h\,E)_t + (\mathcal{F}_e)_x = 0, \tag{7.42}$$

where $h\,E = \dfrac{1}{2}\,h\,u^2 + P_e$ is the energy density and $\mathcal{F} = h\,u\,E + W_p$ is the energy flux. **Find** explicit expressions for P_E and W_p — **what physical interpretation do these two quantities have?**

Part II

Introduce the Lagrangian coordinate $z = \displaystyle\int_{x^*}^{x} h(s,\,t)\,ds$ — where $x^* = x^*(t)$ is a point following the flow, so that $\dfrac{dx^*}{dt} = u^* = u(x^*,\,t)$ — and **show that** the equations take the form

$$v_t - u_z = 0 \quad \text{and} \quad u_t + \left(\frac{1}{2}\,g\,h^2\right)_z = 0. \tag{7.43}$$

These equations remain valid even when hydraulic jumps arise: the first equation expresses volume conservation[35] in Lagrangian coordinates, while the second expresses momentum conservation.[36]

Show that in these coordinates, when hydraulic jumps are absent, the equation for the conservation of the mechanical energy takes the form

$$E_t + (W_p)_z = 0. \tag{7.44}$$

Part III

Show that hydraulic jumps dissipate: mechanical energy is lost at hydraulic jumps — the lost energy, presumably, becoming internal (thermal) energy.[37] *Proceed as follows:*

[35] Note that $v\,dz = v\,h\,dx = dx$, so that v is the volume density.

[36] Note that $u\,dz = h\,u\,dx$, so that u is the momentum density.

[37] You are not asked to show this. The process is quite complicated, and not completely understood. For example: part of the energy goes directly into heat via viscous dissipation by the turbulent eddies generated at the jump. Another part goes into surface energy at the gas bubbles created as air is entrained — but these bubbles eventually disappear, as they move back to the surface or the air in them dissolves into the water.

Step III-1. As pointed out in part II, the equations in (7.43) remain valid at hydraulic jumps. Hence, assume a hydraulic jump with constant speed $D > 0$, moving to the right into fluid at rest — where $h = h_0 > 0$, $v = v_0 = 1/h_0$, and $u = u_0 = 0$. Then the Rankine-Hugoniot jump conditions yield

$$D\,[v] + [u] = 0 \quad \text{and} \quad -\,D\,[u] + \left[\frac{1}{2}\,g\,h^2\right] = 0, \tag{7.45}$$

where the brackets denote the jumps across the shock of the enclosed quantities — e.g.: $[v] = v_0 - v_1$ — and a subscript 1 indicates the value of a variable behind (to the left of) the jump.

Remark 7.7 *There is no loss in generality in assuming a hydraulic jump moving to the right into fluid at rest, because the equations are: (a) Left-right reflection invariant. (b) Galilean invariant.*

Remark 7.8 *Note that z has units of area, hence the shock "speed" D in Lagrangian coordinates has units of area per second. In other words: D is the volume flow per unit width across the jump.*

Remark 7.9 *Notice that, because D is constant, the state behind the hydraulic jump is also constant. Thus the flow considered here is very simple: $(h,\, u) = (h_0,\, 0)$ ahead of the shock ($z > D\,t$) and $(h,\, u) = (h_1,\, u_1)$ behind the shock ($z > D\,t$). The conclusions, however, apply in general, since the behavior of shocks is controlled by the local values of the variables — not their derivatives.*

Furthermore, the *Lax entropy conditions* must apply $\boxed{a_1 > D > a_0}$ — where $a = \sqrt{g\,h^3}$ is the characteristic speed in Lagrangian coordinates. **Show that** this is equivalent to either of

$$h_1 > h_0 \quad \text{or} \quad u_1 > 0. \tag{7.46}$$

Remark 7.10 *In fact, the purpose of this problem is to show that* **the Lax entropy conditions are exactly equivalent to the statement that hydraulic jumps dissipate.** *The solutions to the Rankine-Hugoniot jump conditions that do not satisfy the Lax entropy conditions* create (!) *mechanical energy at the hydraulic jumps. Thus they either would violate conservation of energy, or would have to transform internal (thermal) energy into mechanical energy — thus decreasing the total amount of entropy in the system, violating the second law of thermodynamics.*

Step III-2. As you were asked to show in another problem, in the presence of a shock, equation (7.44) should be modified to

$$E_t + (W_p)_z = -d\,\delta(z - D\,t), \tag{7.47}$$

where $\delta(\cdot)$ is Dirac's delta function, $z = D\,t$ is the position of the shock, and $d = D\,[E] - [W_p]$. The statement that *hydraulic jumps dissipate follows because* $d > 0$ — **show this.** This last equation shows that *a point sink of mechanical energy appears at the location of the jumps.*

Furthermore: **show that** for solutions of the Rankine-Hugoniot jump conditions that *do not* satisfy the Lax entropy condition (hence $h_1 < h_0$ and $u_1 < 0$), $d < 0$. This proves the point in remark **7.10**.

Hint 7.1 *It is convenient to carry the algebra in non-dimensional variables. Use the following non-dimensional variables in your calculations:* $h = (D^{2/3}/g^{1/3})\,\tilde{h}$, $u = D^{1/3}\,g^{1/3}\,\tilde{u}$, $E = D^{2/3}\,g^{2/3}\,\tilde{E}$, *and* $d = D^{5/3}\,g^{2/3}\,\tilde{d}$, *where the variables with tildes have no dimension.*

8 Singularities and characteristics.

8.1 Statement: Singularities in PDE solutions (problem 01).

Consider the following PDE problem, for the real valued function $u = u(x,\,y)$

$$\left(1 + x^{2/3}\right)\,u_x + u_y = 0, \quad \text{with} \quad u(x,\,0) = x \quad \text{for} - \infty < x < \infty. \tag{8.1}$$

Using the method of characteristics, **show that** the solution is defined (everywhere & single-valued) by the (implicit) equation

$$y - f(u,\,x) = 0, \quad \text{where} \quad f(u,\,x) = \int_u^x \frac{ds}{1 + s^{2/3}}. \tag{8.2}$$

This solution has a weak singularity along $x = 0$ — **show this and specify the nature of the singularity:** which derivatives fail to exist? Nevertheless, $x = 0$ is not a characteristic **(show this)**. This does not contradict the statement that weak singularities can exist only along characteristics,[38] because (in this case) the singularity is caused by the coefficients in the equation — which have a singularity along $x = 0$. **Where else does the solution above have a weak singularity?**

8.2 Statement: Steady State Shallow Water (problem 01).

The conservation form of the equations for 2-D shallow water waves over a flat bottom is

$$0 \;=\; h_t + (h\,u)_x + (h\,v)_y, \tag{8.3}$$

[38] Provided the coefficients in the equation are smooth.

$$0 = (h\,u)_t + (h\,u^2 + \frac{1}{2}\,g\,h^2)_x + (h\,v\,u)_y, \tag{8.4}$$

$$0 = (h\,v)_t + (h\,u\,v)_x + (h\,v^2 + \frac{1}{2}\,g\,h^2)_y, \tag{8.5}$$

where h is the fluid depth, u is the x-flow velocity, v is the y-flow velocity, and g is the acceleration of gravity. The steady state (time independent) form of these equations is

$$0 = (h\,u)_x + (h\,v)_y, \tag{8.6}$$

$$0 = (h\,u^2 + \frac{1}{2}\,g\,h^2)_x + (h\,v\,u)_y, \tag{8.7}$$

$$0 = (h\,u\,v)_x + (h\,v^2 + \frac{1}{2}\,g\,h^2)_y. \tag{8.8}$$

Answer the following questions:

1. Under which conditions on $(h,\ u,\ v)$ is (8.6 – 8.8) strictly[39] hyperbolic? Use the **Froude number** $\boxed{F = \sqrt{(u^2+v^2)/(g\,h)}}$ in your answer. You need $F > 0$ to even ask the question — **WHY?**
2. When the characteristic equation has a double root, the system is not hyperbolic.[40] Show this.
   Hint: The system is invariant under rotations. Hence, when computing the eigenvector(s), you can rotate the coordinate system so that $v = 0$ at the point of interest.
3. The system always has (at least) one characteristic, which has a Riemann invariant. Find it.

9 Transformations.

9.1 Statement: Hodograph transformation (problem 01).

The *hodograph transformation* reverses the roles of the independent and dependent variables. It is helpful when dealing with a quasi-linear first order nonlinear P.D.E. such that:

a. The equation is homogeneous: all the terms involve exactly one derivative.

b. The coefficients do not involve the independent variables.

c. The number of dependent variables does not exceed the number of independent variables.

In such cases the transformation can be used to *linearize the P.D.E.* The transformation is, in some vague sense, a generalization to P.D.E. of *separation of variables.* Namely, the solution of the

[39] All the characteristic directions are distinct: the characteristic equation has three distinct roots.

[40] There is only one eigenvector associated with the double root.

nonlinear equation $\dfrac{dy}{dt} = f(y)\,g(t)$ by separation of variables is equivalent to the following process: *(1st)* Introduce a new independent variable by $ds = g\,dt$, which transforms the equation into one with no dependence on the independent variable $\dfrac{dy}{ds} = f(y)$. *(2nd)* Invert the roles of the dependent and independent variables, to obtain the linear equation $\dfrac{ds}{dy} = \dfrac{1}{f(y)}$.

Example 1: scalar 1-st order evolution in one space dimension.

Consider the following nonlinear equation, for the real valued function $u = u(x,\,t)$,

$$u_t + c(u)\,u_x = 0, \quad \text{where } c = c(u) \text{ is some given function.} \tag{9.1}$$

1-a. Do a hodograph transformation, and write the equation for $x = X(u,\,t)$. This should give you a trivial equation. Write the general solution to this equation. How is this solution related to the solution to the initial value problem $u(x,\,0) = f(x)$ to (9.1), obtained by characteristics?

1-b. Do the same for $t = T(u,\,x)$.

Example 2: scalar 1-st order evolution in two space dimensions.

Find a transformation that linearizes the equation

$$u_t + a(u)\,u_x + b(u)\,u_y = 0, \quad \text{where } a = a(u) \text{ and } b = b(u) \text{ are some given functions.} \tag{9.2}$$

Example 3: isentropic Gas Dynamics in one space dimension.

The (inviscid) Euler equations of Gas Dynamics, in mass-Lagrangian coordinates, are as follows

$$v_t - u_\zeta = 0, \quad \text{and} \quad u_t + p_\zeta = 0, \tag{9.3}$$

where v is the specific volume, u is the flow velocity, and $p = p(v)$ is the pressure. Furthermore: $a^2 = -\dfrac{dp}{dv} > 0$ — with $a > 0$ the sound speed in mass-Lagrangian coordinates.

3-a. Do a hodograph transformation, and write the equations that $\zeta = Z(v,\,u)$ and $t = T(v,\,u)$ satisfy. This should give you a system of two, first order, linear equations in Z and T.

3-b. Write the characteristic equations for the system that you obtained in **3-a**.

Recall that the system of equations in (9.3) is equivalent (for solutions without shocks) to the following Riemann invariant form:

$$u \mp b = \text{constant} \quad \text{along the characteristic curves} \quad \frac{d\zeta}{dt} = \pm a, \tag{9.4}$$

where $b = b(v) = \int^v a(s)\, ds$. In particular, the solutions such that one of the Riemann variables $u \mp b$ is identically constant (not just constant along each characteristic) are called *simple waves.*

3-c. What is the relationship of the characteristics for the system that you found in **3-a**, with the Riemann variables $R_\pm = R_\pm(u,\, v) = u \mp b$?

3-d. Assume a simple wave solution to the system in (9.3). What is peculiar about the hodograph map $(\zeta,\, t) \to (u,\, v)$ in this case — what is the image in the $(u,\, v)$ plane? Is there an inverse map $\zeta = Z(v,\, u)$ and $t = T(v,\, u)$, as assumed when doing the hodograph transformation?

9.2 Statement: Gas Dynamics (Eulerian to Lagrangian formulation).

The (inviscid) Euler equations of Gas Dynamics, in one space dimension, have the following form in the laboratory (Eulerian) frame[41]

$$\left.\begin{array}{rcccl} \rho_t & + & (\rho\, u)_x & = & 0 \quad \text{(conservation of mass)}, \\ (\rho\, u)_t & + & (\rho\, u^2 + p)_x & = & 0 \quad \text{(conservation of momentum)}, \\ (\rho\, E)_t & + & (\rho\, E\, u + p\, u)_x & = & 0 \quad \text{(conservation of energy)}, \end{array}\right\} \tag{9.5}$$

where $\rho = \rho(x,\, t)$ is the mass density, $u = u(x,\, t)$ is the flow velocity, $p = p(x,\, t)$ is the pressure, $E = \dfrac{1}{2}\, u^2 + e$ is the energy per unit mass, and e is the internal energy per unit mass — given by an *equation of state* $e = \mathcal{E}(\rho,\, p)$. Of course: x is distance and t is time.

In the special case of adiabatic (constant entropy) motion, we have $0 = T\, dS = de - p\, dv$, where T is the temperature, S is the entropy, and $v = 1/\rho$ is the specific volume. Then the equation of state specifies p as a function of ρ, and the system can be reduced to the first two equations above — the conservation of energy equation is not needed.

Introduce the (mass) Lagrangian coordinate $\zeta = \zeta(x,\, t)$ by

$$\zeta = \int_{x^*}^{x} \rho(s,\, t)\, ds, \tag{9.6}$$

where $x = x^*(t)$ is the position of some (arbitrary) point in the gas — i.e.: $\dfrac{dx^*}{dt} = u(x^*,\, t)$. Then, as long as no vacuum state arises,[42] the transformation from the Eulerian coordinates $(x,\, t)$ to the

[41] In the absence of body forces, such as gravity.

[42] So that $\rho > 0$ everywhere.

(mass) Lagrangian coordinates $(\zeta,\, t)$ has an inverse, given by

$$x = x^* + \int_0^{\zeta} v(z,\, t)\, dz, \tag{9.7}$$

where we think of v as a function of ζ and t inside the integral — i.e.: $v = v(\zeta,\, t)$.

Assume now that no vacuum state arises, and that the solutions are piece-wise smooth (i.e.: shocks, for example, are allowed). Then

1. Prove the formula in (9.7) for the inverse.
2. Let $Z = Z(x,\, t)$ be the function defined by the right hand side in (9.6) — i.e.: $\zeta = Z(x,\, t)$. Show that

   $$Z_x = \rho \quad \text{and} \quad Z_t = -\rho\, u. \tag{9.8}$$

   In particular: $Z_t + u\, Z_x = 0$, so that Z is constant along the particle paths — this shows that, indeed, ζ is a Lagrangian coordinate.
3. Let $X = X(\zeta,\, t)$ be the function defined by the right hand side in (9.7) — i.e.: $x = X(\zeta,\, t)$. Show that

   $$X_\zeta = v \quad \text{and} \quad X_t = u. \tag{9.9}$$

   Again: $\zeta =$ constant should be a particle path, hence its Eulerian coordinate must move at the flow speed.
4. Transform coordinates, from Eulerian to Lagrangian, in the equations given by (9.5). Show that, in Lagrangian coordinates, the equations have the conservation form

   $$\left.\begin{array}{rcrcll} v_t & - & u_\zeta & = & 0 & \text{(conservation of volume)}, \\ u_t & + & p_\zeta & = & 0 & \text{(conservation of momentum)}, \\ E_t & + & (p\,u)_\zeta & = & 0 & \text{(conservation of energy)}. \end{array}\right\} \tag{9.10}$$

   Notice how much more compact than in Eulerian coordinates the equations are.
5. Derive directly, using conservation arguments, the equations in (9.10).

Hints regarding item 4. *It is when doing item* **4** *that you will have to pay particular attention to the fact that derivatives may fail to exist in the classical sense.* The usual ways in which coordinate transformations are carried will not remain valid when discontinuities in the solution arise. I encourage you to (first) carry the transformation in the usual way (assuming that all the functions involved have derivatives), to convince yourself that the calculations cease to make sense when discontinuities arise. For example, what is the meaning of stuff like $u\,p_x$ when both u and p have a discontinuity at some point $x = s$? In this case p_x will have a Dirac's delta function contribution at $x = s$ and, since u has no unique value at $x = s$, the product $u\,p_x$ has no meaning.

In order to avoid the difficulties pointed out in the prior paragraph (when discontinuities in the solution are present) you should *go back to the integral formulation of the conservation laws,*[43] *and transform them directly.* Namely, instead of (9.5), use:

$$\begin{aligned}
\frac{d}{dt}\left(\int_a^b \rho\,dx\right) &= \Big((u-\dot{a})\,\rho\Big)_{x=a} - \Big((u-\dot{b})\,\rho\Big)_{x=b} && \text{(cons. of mass)},\\
\frac{d}{dt}\left(\int_a^b \rho\,u\,dx\right) &= \Big((u-\dot{a})\,\rho\,u+p\Big)_{x=a} - \Big((u-\dot{b})\,\rho\,u+p\Big)_{x=b} && \text{(cons. of momentum)},\\
\frac{d}{dt}\left(\int_a^b \rho\,E\,dx\right) &= \Big((u-\dot{a})\,\rho\,E+p\,u\Big)_{x=a} - \Big((u-\dot{b})\,\rho\,E+p\,u\Big)_{x=b} && \text{(cons. of energy)},
\end{aligned}$$

for any interval $a = a(t) < x < b = b(t)$ — and transform these into Lagrangian coordinates. You should find out that the conservation of mass transforms into a trivial equation in Lagrangian coordinates. On the other hand, the conservation of volume, which is trivial in Eulerian coordinates:

$$\frac{d}{dt}\left(\int_a^b dx\right) = -\dot{a}+\dot{b},$$

yields a non-trivial equation in Lagrangian coordinates.

10 Wave Equations.

10.1 Statement: Wave equations (problem 01).

Consider an elastic (homogeneous) string under tension, tied at one end, initially at rest, and forced by a (small amplitude) harmonic shaking of the other end. To simplify the situation, assume that all the motion is restricted to happen in a plane.

After a proper adimensionalization, the situation is modeled by the mathematical problem below for the wave equation in 1-D — where $u = u(x,\,t)$ is the displacement from equilibrium of the string.

$$u_{tt} - u_{xx} = 0, \quad \text{for} \quad 0 < x < 1, \quad \text{and} \quad t > 0, \tag{10.1}$$

[43] Which remain valid even for discontinuous solutions.

with *initial data* $u(x,\,0) = u_t(x,\,0) = 0$, and *boundary conditions*

$$u(0,\,t) = 1 - \cos(\omega\,t) \quad \text{and} \quad u(1,\,t) = 0. \tag{10.2}$$

FIND the solution to this problem, for the times $0 < t \leq 4$. Furthermore: note that the solution, while making sense in the classical sense (no need to invoke generalized function derivatives), is not infinitely differentiable. There are certain lines along which "singularities" occur. **FIND** these lines of singularity, and describe what the situation is along them (nature of the singularities) — the lines are, of course, characteristics. **FINALLY:** does anything special happen if $\omega = \pi$?

10.2 Statement: Wave equations (problem 02).

Consider an elastic (homogeneous) string under tension, undergoing small amplitude oscillations, and assume that all the motion is restricted to happen in a plane. Under these conditions, and after a proper adimensionalization, the displacements $u = u(x,\,t)$ from equilibrium of the string can be shown to satisfy the 1-D wave equation

$$u_{tt} - u_{xx} = 0, \tag{10.3}$$

to which appropriate *initial data* and *boundary conditions* must be applied to determine a unique solution.

In the lectures we showed that the second order (in space and time) equation in (10.3) is equivalent to a system of two first order equations. We did this by introducing the variables $v = u_t$ and $w = u_x$, for which it can be seen that

$$v_t - w_x = 0 \quad \text{and} \quad w_t - v_x = 0 \tag{10.4}$$

apply — the first equation is (10.3) and the second follows from equality of cross-derivatives. On the other hand, the second equation in (10.4) guarantees that there is a u such that $v = u_t$ and $w = u_x$, and then the first equation yields (10.3).

Consider now the situation where the string is attached to an (homogeneous) "elastic bed", instead of being free in space.[44] In this case, in addition to the forces caused by the tension in the string,

[44] For example: imagine a ribbon made of some elastic material, with one edge attached to a rigid surface, the other edge attached to the string, and thin enough that we can ignore its mass.

forces are generated by the bed — which are functions of the displacements u only. Thus the governing equation above in (10.3) must be modified to

$$u_{tt} - u_{xx} + g(u) = 0, \tag{10.5}$$

where g characterizes the elastic response by the bed. If Hooke's law applies, then $g = \kappa\, u$ — for some elastic constant $\kappa > 0$.

By introducing appropriate variables, **SHOW THAT** *the second order equation in (10.5) is equivalent to a first order system of two equations in two unknowns functions.*

Hint: because the function u appears in equation (10.5), the trick that we used for (10.3) does not work for (10.5). If you introduce $v = u_t$ and $w = u_x$ as new variables, you will also have to keep u, and then you will end up with three variables (not two). Instead, try introducing as a new variable an appropriate combination of u_t and u_x.

Note that the approach that you develop here should work for any $g = g(u)$. In particular, for $g \equiv 0$ it will give you a *different* (from the one used in the lectures) way to show that (10.3) is equivalent to a system of two first order equations.

10.3 Statement: Wave equations (problem 03).

This problem investigates the issue of the characteristics as the places where "weak" singularities of the solutions can occur — where by "weak" singularities we mean lack of smoothness in the solutions which is not strong enough to destroy their meaning as classical solutions.

EXAMPLE: consider the linear first order scalar equation for $u = u(x,\, y)$

$$a\, u_x + b\, u_y + c\, u = d, \tag{10.6}$$

where $a = a(x,\, y)$, $b = b(x,\, y)$, $c = c(x,\, y)$, and $d = d(x,\, y)$, are some given smooth functions, with $a^2 + b^2 \neq 0$. Consider now a function $u = u(x,\, y)$ and a curve $\phi(x,\, y) = 0$ — where ϕ is smooth and has a non-zero gradient, such that $u = u(x,\, y)$ is continuous and

1. u has continuous partial derivatives where $\phi \neq 0$, which have a continuous limit on each side of the curve — in other words: the graph of u is a "nice" surface, except that it has a crease along the given curve. *Derivatives in directions parallel to the curve exist and are continuous*

everywhere, while derivatives in directions that cross the curve have a simple discontinuity as the curve is crossed.

2. On each side of the curve, u satisfies equation (10.6). Because of item **1**, the limits of the solution (as the curve is approached on each side) on the curve, also satisfy the equation.

Such a u is a solution to equation (10.6) in the "classical" sense, but it has a lack of smoothness across the curve $\phi = 0$ — which is as strong as it can be, while still allowing a solution in the classical sense. *Question: are there any restrictions on what the curve $\phi = 0$ can be?*

In order to answer the question in the prior paragraph, we introduce a local (curvilinear) coordinate system such that ϕ is one of the coordinate functions — this can always be done. So, let $\psi = \psi(x, y)$ be a smooth function with non-zero gradient such that $\nabla\phi$ and $\nabla\psi$ are not co-lineal, and re-write the equation using ϕ and ψ as independent variables. Then

$$d = (a\,\phi_x + b\,\phi_y)\; u_\phi + (a\,\psi_x + b\,\psi_y)\; u_\psi + c\,u, \tag{10.7}$$

which should apply for $\phi > 0$ and $\phi < 0$, with continuous limits as $\phi \to 0$ from each side. Furthermore, from item **1** it follows that u_ψ is continuous everywhere, while u_ϕ has a simple discontinuity at $\phi = 0$. Thus taking the limit (from both sides $\phi > 0$ and $\phi < 0$) as $\phi \to 0$ of the equation, and then taking the difference of these two limits, we obtain

$$0 = (a\,\phi_x + b\,\phi_y)\; [u_\phi] \quad \text{along} \quad \phi = 0, \tag{10.8}$$

where $[u_\phi]$ denotes the (non-zero) jump in u_ϕ across the curve. Hence we obtain the following equation that must be satisfied by the curve

$$0 = a\,\phi_x + b\,\phi_y \quad \text{along} \quad \phi = 0. \tag{10.9}$$

Since $\phi_x\,dx + \phi_y\,dy = 0$ along the curve, it follows that

$$a\,dy = b\,dx, \tag{10.10}$$

which is equivalent to the (parametric) equation for the characteristics obtained in the lectures (by other means); namely: $\dfrac{dx}{ds} = a$ and $\dfrac{dy}{ds} = b$.

THESE ARE YOUR TASKS IN THIS PROBLEM:

PART I ....

The arguments above appear to impose no restrictions on the curve $\phi = 0$ if the singularities in u appear at higher order. This is **not true.** For example, assume that u has continuous derivatives up to second order, except that the second derivatives have simple discontinuities across $\phi = 0$. Thus, *it is the graph of (say) u_x that has a crease along the curve.* **SHOW** then that *the curve must be a characteristic.* **Hint:** Consider the equation that u_x satisfies.

PART II ....

Consider the linear second order scalar equation for $u = u(x, y)$

$$a\,u_{x\,x} + 2\,b\,u_{x\,y} + c\,u_{y\,y} = d, \tag{10.11}$$

where $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$, and $d = d(x, y)$, are some given smooth functions, with $a\,c - b^2 \neq 0$. Using an argument similar to the one in (10.6 – 10.10), **FIND** the curves across which the solutions to the equation can have "weak" discontinuities — i.e.: the **characteristics.** In particular:

II-1. Under which conditions on the coefficients $a, b, \ldots$ do such curves exist?

II-2. What are the curves in the case $u_{x\,x} - u_{y\,y} = 0$?

II-3. What happens in the case $u_{x\,x} + u_{y\,y} = 0$?

Hint: In this case you have to assume that it is the second partial derivatives of u that have simple discontinuities across the curve $\phi = 0$.

11 Weak solutions and generalized functions.

11.1 Statement: Weak solutions (problem 01).

Let $x \to f(x)$, $-\infty < x < \infty$, be a piece-wise C^1 real valued function. Namely: there is a (finite) number of points $-\infty < x_1 < x_2 < \ldots < x_N < \infty$ at which f is not defined, and

1. $f = f(x)$ has a continuous derivative in each of the intervals $x_n < x < x_{n+1}$, $0 \leq n \leq N$, where $x_0 = -\infty$ and $x_{N+1} = \infty$.

2. At each point x_n, $1 \leq n \leq N$, both the left $f_n^- = \lim_{x \to x_n,\, x < x_n} f(x)$ and right $f_n^+ = \lim_{x \to x_n,\, x > x_n} f(x)$ limits are defined and finite. The derivative f' has the same property.

Using the definition of a generalized function derivative, **show that**

$$f'(x) = f'_f(x) + \sum_{n=1}^{N} [f]_n \, \delta(x - x_n), \tag{11.1}$$

where δ is the Dirac's delta function, $[f]_n = f_n^+ - f_n^-$ is the jump in the function at x_n, and f'_f is the "usual" derivative of f — which is only defined for $x \neq x_n$, and is piece-wise continuous.

Note: Assume that your test functions ϕ vanish outside some finite interval, and have infinitely many derivatives. That is $\phi \in C_0^\infty$.

11.2 Statement: Weak solutions (problem 02).

Let $\rho = \rho(x,\, t)$ be a piece-wise C^1 real valued function defined on space-time. Specifically: assume that there is a smooth curve $x = x_s(t)$ such that ρ is defined everywhere but on this curve, and

1. $\rho = \rho(x,\, t)$ has a continuous (partial) derivatives for $x < x_s$ and $x > x_s$.
2. $\rho = \rho(x,\, t)$ and its partial derivatives have both left and right (finite and continuous) limits along the curve $x = x_s(t)$.

For any continuous function $h = h(\rho)$, define h^- and h^+ as the left and right limits, respectively, of h along the curve $x = x_s(t)$. Namely: $h^- = \lim\limits_{x \to x_s,\, x < x_s} h(\rho)$ and $h^+ = \lim\limits_{x \to x_s,\, x > x_s} h(\rho)$. Furthermore, let $[h] = h^+ - h^-$ be the jump in h across the curve.

Using the definition of generalized function derivatives, **show that:**
For any continuously differentiable functions $f = f(\rho)$ and $g = g(\rho)$,

$$f_t + g_x = (f_t + g_x)_f + (-\sigma\, [f] + [g])\; \delta\left(x - x_s(t)\right), \tag{11.2}$$

where $(\cdot)_f$ is used to indicate the standard derivatives — defined for $x \neq x_s$ only, δ is the Dirac delta function, and $\sigma = dx_s/dt$.

Note 1: Assume that your test functions $\phi = \phi(x,\, t)$ have infinitely many derivatives and vanish outside some bounded region in space-time. That is $\phi \in C_0^\infty$.

Note 2: The curve given by $x = x_s(t)$ need not be monotone. In other words, $\sigma = \sigma(t)$ can vanish, switch signs, etc.

12 Well and ill posed problems.

12.1 Statement: Ill posed laplacian problem.

Consider the following problem involving Laplace's equation

$$u_{xx} + u_{yy} = 0, \tag{12.1}$$

on the strip $0 < x < 2\,\pi$ and $-\infty < y < \infty$:

Given $u(0, y) = f(y)$ and $u_x(0, y) = g(y)$, where f and g are smooth periodic functions, determine $u(2\,\pi, y) = h(y)$.

Show that this is an ill-posed problem.

Hint: Consider what happens with high frequency perturbations.

12.2 Statement: Laplace equation (problem 01).

Consider a thin, homogeneous, heat conducting sheet, insulated on the top and the bottom. If $T = T(x,\, y,\, t)$ is the temperature in the sheet, then the conservation of heat (and Fick's law) leads to the heat equation — which in non-dimensional units has the form

$$T_t = \Delta\, T = T_{xx} + T_{yy}. \tag{12.2}$$

Let Ω be the region of space occupied by the sheet, and assume that along the boundary $\partial\Omega$ of this region the heat flux is known and given by some function, say: $F = F(s)$ per unit length (where s is the arc-length along $\partial\Omega$).

The problem to be solved is then (12.2) inside Ω, with the boundary conditions on $\partial\Omega$

$$\partial_n T = \hat{n} \cdot \nabla T = F(s), \tag{12.3}$$

where $\hat{n}$ is the unit outside normal to $\partial\Omega$. In particular, for *steady state*, we have Laplace's equation in Ω

$$0 = \Delta\, T = T_{xx} + T_{yy}, \tag{12.4}$$

with the Neumann conditions in (12.3).

1. Show that there is an integral condition that F must satisfy if the problem (12.3 – 12.4) has a solution. **Hint:** Gauss theorem.

2. Give a physical interpretation to the condition in **1.** Why do you need it, and what happens when it is not satisfied?

3. The solution to (12.3 – 12.4), if there is one, is determined only up to an arbitrary additive constant. How would you determine this constant, and what is it related to — i.e.: knowledge of what physical quantity gives it to you?

12.3 Statement: PDE Blow Up.

The purpose of this problem is to investigate an example where smooth solutions to a PDE cease to exist after finite time. In the lectures (see the remark below) we considered the problem:

$$u_t + u u_x = 0\,, \quad \text{with initial data} \quad u(x,0) = F(x)\,, \tag{12.5}$$

where

- $u = u(x,t)$ is a function of x (space) and t (time).
- u_t and u_x denote the partial derivatives of u with respect to t and x.

We saw that: **The solution to problem (12.5) ceases to exist at a finite time (the derivatives of u become infinite and, beyond that, u becomes multiple valued), whenever dF/dx is negative somewhere.** (12.6)

Consider now the problem:

$$u_t + u u_x = -u\,, \quad \text{with initial data} \quad u(x,0) = F(x)\,. \tag{12.7}$$

Show that the solution to this second problem ceases to exist at a finite time, provided that $dF/dx < C < 0$, where C is a finite (non-zero) constant. Again, what happens is that the derivatives become infinite. Calculate C.

<u>**Hint**</u>**:** *To show the result (quoted above) in class, we used **two** (related) approaches, both involving the characteristic curves. In the first we obtained the exact solution to the problem (in implicit form) and from that we obtained the result. The second approach went straight to the point and showed*

that u_x had to "blow up" on some characteristic curve. Both techniques will work here too, but the second approach is much simpler (for the purposes of what you are being asked to show).

Remark: *In order to show the result in (12.6), we proceeded in two ways.*

(1-st). Introduce the characteristic curves in space-time, which — for equation (12.5), are defined by $\dfrac{dx}{dt} = u(x,t)$. Equation (12.5) then yields $\dfrac{du}{dt} = 0$ along these curves, so we can write

$$\left.\begin{array}{rclcrcl} \dfrac{dx}{dt} &=& u & \text{and} & x(0) &=& \zeta \\ \dfrac{du}{dt} &=& 0 & \text{and} & u(0) &=& F(\zeta) \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} x &=& \zeta + F(\zeta)t \\ u &=& F(\zeta) \end{array}\right\} \tag{12.8}$$

for each characteristic, which we label by the value x takes for $t = 0$ — namely: ζ. Thus, for this problem, the characteristics are straight lines in space-time, and along each of them u is constant. Clearly, if the initial data F is decreasing anywhere,[45] then there will be characteristics that cross somewhere in space-time, for a finite time $t > 0$. But then the solution there would be multiple valued! The earliest time at which this happen would correspond to two infinitesimally distant (i.e. ζ and $\zeta + d\zeta$) characteristics crossing, at which point the drivative u_x would become $-\infty$.

Equation (12.8) has the following interpretation — in terms of what the shape of the solution (i.e. $y = u(x,t)$, for each fixed time) does as a function of time. Each value of u propagates at its own speed, that happens to be u. Thus, in places where u is decreasing, the shape steepens (values behind catch-up to the values ahead) and, eventually, an infinite slope arises at some point.

(2-nd). Introduce the characteristic curves, but now directly write an equation for how u_x changes along each characteristic. By taking the x partial derivative of equation (12.5), it is easy to see that

$$\frac{dw}{dt} + w^2 = 0, \quad \text{where} \quad w = u_x. \tag{12.9}$$

Along any characteristic where w starts negative, it reaches $-\infty$ in a finite time. Hence (12.6) follows.

THE END.

[45] Namely: $\dfrac{dF}{dx} < 0$ someplace.