Lecture 2
Complex Analysis

A Complex Numbers and Complex Variables

In this chapter we give a short discussion of complex numbers and the theory of a function of a complex variable.

Before we get to complex numbers, let us first say a few words about real numbers.

All real numbers have meanings in the real world. Ever since the beginning of civilization, people have found great use of real positive integers such as 2 and 30, which came up in, as an example, the sentence “my neighbor has two pigs, and I have thirty chickens”. The concept of negative real integers, say -5, is a bit more difficult, but found its use when a person owed another person five ounces of silver. It was also natural to extend the concept of integers to numbers which are not integers. For example, when six persons share equally a melon, the number describing the fraction of melon each of them has is not an integer but the rational number 1/6. The need for other real numbers was found as mathematicians pondered the length of the circumference of a perfectly circular hole with unit radius. This length is a real number which can be expressed neither as an integer nor as a ratio of two integers. This real number is denoted as π, and is called an irrational number. Another example of irrational number is e.

Each of the real numbers, be it integral, rational or irrational, can be geometrically represented by a point on an infinite line, and vice versa..

When we add, subtract, multiply or divide two real numbers, the outcome is always a real numbers. Thus the root of the linear equation

$ax + b = c$,

with a, b and c real numbers, is always a real number. This means that if we restrict ourselves to making linear algebraic operations of real numbers, the result that comes out is invariably found as a real number. Thus the real numbers form a complete system under linear algebraic operations.

But as soon as we get to non-linear operations, the system of real numbers becomes inadequate by itself. For example, the roots of

$x^2 = -1$

cannot be expressed as a real number, and we must use our imagination denoting a root as i. As we all know, the number i is the imaginary number. While we have gotten to be comfortable with the number i ever since our high school days, Gauss once remarked that the “ true metaphysics” of i was “hard”.

The number

$\alpha = a + ib$,

where a and b are real numbers, is called a complex number. The numbers a and b are called the real part and the imaginary part of α, respectively. While complex numbers do not have direct meanings in the real world, we shall see that, as we allow ourselves wander into the never-never land of complex numbers, we find powerful ways to deal with real problems.

The complex variable z is denoted by

$$z = x + iy,$$

where x and y are real variables and

$$i^2 = -1.$$

The complex conjugate of z will be denoted as

$$z^* = x - iy.$$

The variable z can be represented geometrically by the point (x, y) in the Cartesian two-dimensional plane. In complex analysis, this two-dimensional plane is called the complex plane. The x axis is called the real axis, and the y axis in this plane is called the imaginary axis. Let r and θ be the polar coordinates, i.e.,

$$x = r\cos\theta, y = r\sin\theta\,(r \geq 0),$$

where θ can be chosen to be between 0 and 2π.

In the the polar coordinates, z is

$$z = r(\cos\theta + i\sin\theta). \tag{2.1}$$

The Euler's formula says

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{2.2}$$

Incidentally, (2.2) shows that, when θ is real, $\cos\theta$ and $\sin\theta$ are the real part and the imaginary part of $e^{i\theta}$, respectively. For example, the complex num Some useful special cases of (2.2) are

$$e^{i\pi} = -1,$$

which wraps up . In particular,

$$e^{2i\pi} = 1,\ \ e^{i\pi} = -1,$$

which relate the transcendental numbers e, π and i.

By (2.2) we have

$$z = re^{i\theta}. \tag{2.3}$$

This is known as the polar form of z. We call r the absolute value of z, which is also expressed as $|\ z\ |$, and θ the argument of z.

The argument of z is defined modulo an integral multiple of 2π, as is obvious geometrically. Indeed, the polar form (2.3) can be written as

$$z = re^{i(\theta+2n\pi)}. \tag{2.4}$$

The polar form is particularly convenient to use for carrying out the operations of multiplication or division of complex numbers. Let

$$z_1 = r_1e^{i\theta_1},\ \ z_2 = r_2e^{i\theta_2},$$

then

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)},\ \ z_1/z_2 = (r_1/r_2)e^{i(\theta_1-\theta_2)}.$$

These operations are more cumbersome to carry out if we express the complex numbers with the Cartesian form.

Needless to say, using the polar form to do multiplication and division of more factors of complex numbers is even more laborsaving. In particular, we have

$$z^n = r^ne^{in\theta} = r^n(\cos n\theta + i\sin n\theta).$$

What about the power function z^a, where a is a number which is not an integer? We have

$$z^a = \left[re^{i(\theta+2n\pi)}\right]^a = r^ae^{i\theta a}e^{i2n\pi a}.\ \ (n = 0, \pm1, \pm2, \cdots),$$

which shows that z^a has infinitely many values. Exceptions occur when a is a rational number. Consider for example $z^{1/2}$. Setting $a = 1/2$ and $z = 1$ in the expression above, we get

$$\begin{aligned} 1^{1/2} &= e^{in\pi} = 1,\ \ n \text{ even}, \\ &= -1,\ \ n \text{ odd}, \end{aligned}$$

which is the familiar result that the square-root of unity is ±1. More generally,

$$z^{n/m} = r^{n/m}e^{in\theta/m}e^{i2\pi kn/m},\ \ (k = 0, 1, \bullet\bullet\bullet, m-1),$$

where n and m are integers with no common factors.

B Analytic Functions

A complex-value function $f(z)$ is said to be analytic in a region R in the complex z-plane if the limit

$$\lim_{\Delta z\to0}\frac{\Delta f}{\Delta z} \qquad \#$$

exists for every point z in R, where

$\Delta f = f(z + \Delta z) - f(z)$.

The limit above, if it exists, is called the derivative of $f(z)$. The function $f(z)$ is said to be analytic at z_0 if it has a derivative in a neighborhood of z_0.

While (2.5) resembles the definition of the derivative of a function of a real variable x

$$f'(x) = \lim_{\Delta x\to0}\frac{\Delta f}{\Delta x},$$

there is actually a substantive difference between them. The point is that Δz has both a real part and an imaginary part, i.e., $\Delta z = \Delta x + i\Delta y$. Therefore, if $f(z)$ is to have a derivative, the limit of (2.5) is required to exist for any Δx and Δy, as long as both of them go to zero. There is no restriction, for example, on the ratio of $\Delta y/\Delta x$, which may take any value. This is a strong condition on the function $f(z)$.

A strong condition has strong consequences. Let

$$f(z) = u(x,y) + iv(x,y),$$

where u and v are the real and the imaginary part of $f(z)$. Then the expression in (2.5) is

$$\lim_{\Delta z\to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}, \tag{2.6}$$

where

$\Delta u = u(x+\Delta x,\ y+\Delta y) - u(x,y),$

and similarly for Δv. We first consider the limit of (2.6) with Δz real, i.e., $\Delta z = \Delta x$. Then the limit of (2.6) is equal to

$$\lim_{\Delta x\to 0} \frac{\Delta u + i\Delta v}{\Delta x}\bigg|_{y\,\text{fixed}} = u_x + iv_x, \tag{2.7}$$

where u_x, for example, is the partial derivative of u with respect to x. Next we consider the limit (2.6) with Δz purely imaginary, i.e., $\Delta z = i\Delta y$. We have

$$\lim_{\Delta y\to 0} \frac{\Delta u + i\Delta v}{i\Delta y}\bigg|_{x\,\text{fixed}} = \frac{(u_y + iv_y)}{i}. \tag{2.8}$$

If $f(z)$ has a derivative, the expressions of (2.7) and (2.8) are the same by definition. This requires that

$$u_x = v_y,\ u_y = -v_x. \tag{2.9}$$

The equations in (2.9) are known as the Cauchy-Riemann equations which the real part and the imaginary part of an analytic function must satisfy.

While we have only required that the limit of (2.5) is the same with Δz either real or imaginary, it is straightforward to prove that this limit is the same for any complex Δz when the Cauchy-Riemann equations are obeyed.

Problem for the Reader:

Is the function $f(z) = zz^*$ analytic?

Answer

For this function,

$$u(x,y) = x^2 + y^2,\ v(x,y) = 0.$$

We have

$$u_x = 2x,\ u_y = 2y,\ v_x = v_y = 0.$$

Thus the Cauchy-Riemann equations are not satisfied except at the origin, which is a point but not a region. Since the derivative of the function exists for no region of z, it is not analytic anywhere.

Next we give a few examples of functions which are analytic. The power function z^n with n an integer is analytic. While this may appear obvious to many of you, let us give it a proof. We have, by using the binomial expansion,

$$\lim_{\Delta z\to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z} = \lim_{\Delta z\to 0} \frac{nz^{n-1}\Delta z + \bullet\bullet\bullet}{\Delta z},$$

where the terms unexhibited are at least as small as the square of Δz. The limit above exists for all Δz and is equal to nz^{n-1}, the way we remember it from calculus. Thus the derivative of the power function z^n exists for all values of z, and this function is analytic for all values of z, or an entire function of z.

Since the power function z^n is analytic, so is the linear superposition of a finite number of power functions. And so is an absolutely convergent sum of power functions. Conversely, a function analytic at a point z_0 always has a convergent Taylor series expansion around z_0 (homework problem 7).

C The Cauchy Integral Theorem

The contour integral

$$I = \int_c f(z)dz$$

where c is a contour in the complex plane, is defined to be

$$\int_c (u+iv)(dx+idy) = \int_c (udx - vdy) + i\int_c (udy + vdx). \qquad \#$$

We note that the two integrals on the right side of (2.12) are line integrals in the two-dimensional plane.

An example of a line integral is the work done by a force. As we know, if A and B are two points in the $x-y$ plane, the work done in moving a particle from A to B along a path c against the force

$$\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$$

is equal to the line integral

$$\int_c (Mdx + Ndy).$$

We also recall that if $\vec{F}$ is a conservative force, i.e., if there exists a potential V such that

$$\vec{F} = -\vec{\nabla}V,$$

then the work done is independent of the path. To say this more precisely, let the potential V exist in a region R in the two-dimensional plane, then

$$\int_{c_1} (Mdx + Ndy) = \int_{c_2} (Mdx + Ndy),$$

provided that c_1 and c_2 are two curves with the same endpoints and both lie inside R.

If the potential V exists, we have

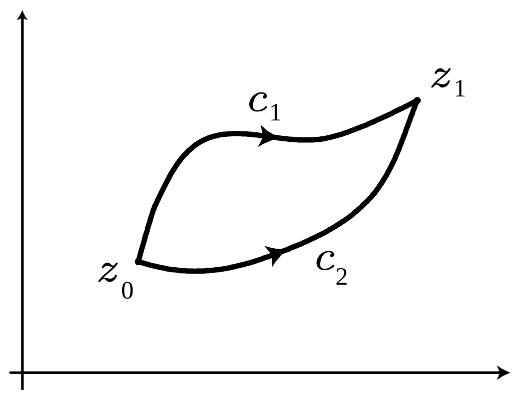

Figure 2.1.

$$M = -V_x,\ N = -V_y,$$

and hence

$$M_y = N_x. \qquad (2.13)$$

The converse is indeed also true: if (2.13) holds in a region R, then the force is the gradient of a potential.

Now for the first line integral in (2.12), M is u and N is $-v$. Thus the condition (2.13) for this line integral is the second Cauchy-Riemann equations. For the second line integral in (2.12), M is v and N is u . Thus the condition (2.13) for this line integral is the first Cauchy-Riemann equations. The contour integral I in (2.12) is therefore path independent if $f(z)$ is analytic. More precisely, let c_1 and c_2 be two curves, both join the lower endpoint z_0 to the upper endpoint z_1 in the complex z-plane, and both lie inside the region R where $f(z)$ is analytic. Then we have

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz. \qquad (2.14)$$

Equation (2.14) tells us that we may deform the contour c_1 to the contour c_2, where c_1 and c_2 have the same endpoints, provided that $f(z)$ is analytic in the region lying between c_1 and c_2.

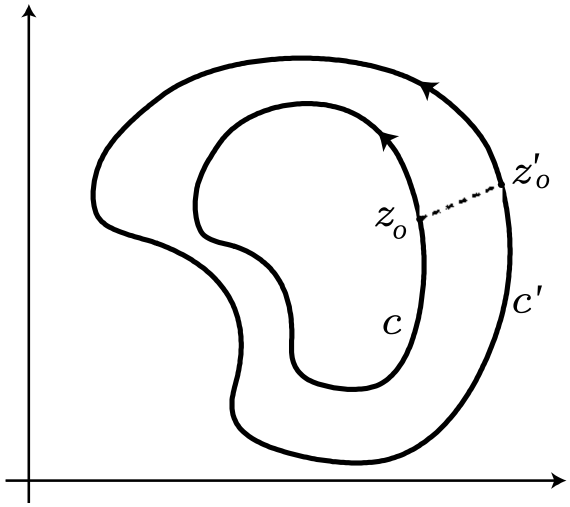

Figure 2.2.

The contours c_1 and c_2 in (2.14) are open contours. We shall extend (2.14) to closed contours. Let c and c' be closed contours of the same sense of direction, i.e., either both counterclockwise or both clockwise, and that there are no singularities of $f(z)$ between c and c'. We choose a point $z_0$on c and think of the closed contour c as a contour joining the point z_0 to itself. Let us draw a line joining z_0 to a point z_0' on c', forming a bridge between c and c'. Then we may think of c' as another contour joining z_0 to itself. This is because c' can be considered to be the contour which begins at z_0, crosses the bridge to z_0', and follows c' to return to z_0', then crosses the bridge in the reverse direction to finally come back to z_0. As the bridge is crossed twice in opposite directions, the two contour integrals associated with the contour of the bridge cancel each other. Therefore, c' can also be considered as a closed contour joining z_0 to itself, and by (2.14) have

$$\int_c f(z)dz = \int_{c'} f(z)dz. \tag{2.15}$$

Equation (2.15) says that the contour c can be deformed into c' provided that $f(z)$ is analytic in the region lying between c and c'.

Let us go from z_0 to z_1 along contour c_1 in Fig. 2-1, then go from z_1 back to z_0 along $-c_2$, which is c_2 in the reverse direction. The contour $c = c_1 - c_2$ is a closed contour. Thus (2.14) can be written as

$$\int_c f(z)dz = 0 \tag{2.16}$$

provided that $f(z)$ is analytic in a region R and c is a closed contour c inside R. Equation (2.16) is the important Cauchy integral theorem.

Next we consider the integral

$$I_n = \int_c \frac{dz}{(z - z_0)^n},$$

where c is a closed contour in the counterclockwise direction and n is a positive integer. The integrand blows up at $z = z_0$, and is said to have a singularity at z_0. More generally, if a single-value function $f(z)$ is not analytic at point z_0, then we say that $f(z)$ has a singularity at z_0.

If c does not enclose z_0, I vanishes by Cauchy's integral theorem.

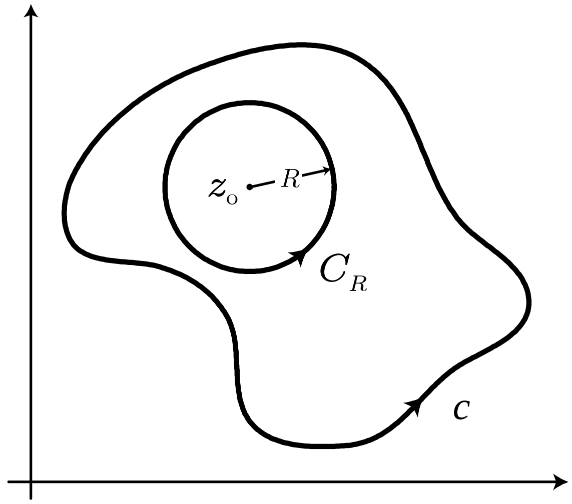

Figure 2.3.

And if c encloses z_0, we may deform the contour into the circle C_R without crossing any singularity of the integrand, where C_R is the circle the center of which is z_0 and the radius of which is R.

Now if z is on C_R,

$$|z - z_0| = R,$$

and hence

$$z - z_0 = e^{i\theta}R.$$

From this polar form of $(z - z_0)$, we get

$$dz = ie^{i\theta}Rd\theta.$$

Thus we have

$$I_n = \frac{i}{R^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta} d\theta.$$

The integral above is easily calculated. Indeed, we have

$$\int_0^{2\pi} e^{i(1-n)\theta} d\theta = 2\pi, \ n = 1,$$
$$= 0, \ n \neq 1.$$

Thus we conclude that, if z_0 is inside the closed counterclockwise contour c, we have

$$I_n = 2\pi i, \ n = 1,$$
$$= 0, n \neq 1. \tag{2.17}$$

From (2.17), we find that if the only singularity $f(z)$ has in the region enclosed by the closed coutour c is located at z_0, and if $f(z)$ is approximately

$$a_{-1}/(z - z_0)$$

as z is near z_0, we have

$$\int_c f(z)dz = 2\pi i a_{-1}. \tag{2.26}$$

This is known as the Cauchy residue theorem. Eq. (2.26) is actually true as long as $f(z)$ has an isolated singularity at z_0. (See the textbook for a more complete discussion.) The coefficient a_{-1} is known to be the residue of $f(z)$ at z_0, which we shall denote as Res(z_0). If the contour is clockwise, the integral will be equal to the negative of $2\pi i$ times the residue.

This formula is one of the most useful formulae in complex analysis. It tells us that the value of an integral over a closed contour can be obtained by simply evaluating the residue of its integrand.

If the contour c encloses more than one singularities of $f(z)$, we replace the right side of (2.26) by the sum of residues of $f(z)$ at these singularities.

Before we close this section, let us show how to evaluate efficiently the residue of $f(z)$ at z_0 where the function has a pole of the first order, which is called a simple pole. If the singularity of $f(z)$

at z_0 is a simple pole,

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \bullet\bullet\bullet.$$

Thus the residue of $f(z)$ at z_0 is equal to

$$\mathrm{Res}(z_0) = \lim_{z \to z_0}(z - z_0)f(z). \tag{2.27}$$

D Evaluation of Real Integrals

The Cauchy residue theorem provides us with a tool to evaluate a number of real integrals, the integrands of which are functions of a real variable and the integration is over real values of the variable. Some of these integrations are difficult to carry out in closed form with the methods provided by calculus. We shall show that, by going into the never-never land of the complex plane, sometimes we can find the closed forms of these integrals.

As an example, let us consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}. \tag{2.28}$$

This integral can be evaluated exactly. We have

$$I = \tan^{-1}x|_{-\infty}^{\infty} = \pi.$$

We shall reproduce this result by using the Cauchy residue theorem. We regard this integral as a contour integral over the real axis of the complex plane. But we cannot as yet apply the Cauchy residue theorem to it, as the real axis is not a closed contour. Let us think of the real axis as the contour from $-R$ to R along the real axis, in the limit as R approaches infinity. We add to this contour the counterclockwise semicircle in the upper half-plane with the origin as the center and R the radius, and get a closed contour which we shall call c. As we shall see, the integral over the semicircle vanishes in the limit of $R \to \infty$. Thus the integral of (2.28) is equal to the integral over c. Since c is a closed contour we may apply the Cauchy residue theorem to the integral. The only singularity of the integrand enclosed by c is $z = i$. Thus we get

$$I = 2\pi i \mathrm{Res}(i) = 2\pi i \frac{1}{2i} = \pi,$$

which is the correct result.

To finish the argument let us show that the contribution of the semicircle is zero in the limit $R \to \infty$. If z is a point on the semicircle,

$$z = e^{i\theta}R, \ \ 0 \le \theta \le \pi.$$

When R is very large, the integrand $1/(1 + z^2)$ is approximately equal to $1/z^2$, the magnitude of which is $1/R^2$. We also have

$$dz = ie^{i\theta}Rd\theta. \tag{2.29}$$

Thus we have

$$\int_{C_R} \frac{dz}{1 + z^2} \approx \int_0^{\pi} \frac{Rid\theta}{R^2 e^{2i\theta}},$$

where C_R is the semicircle in the upper-half plane. In the limit $R \to \infty$, the integral above vanishes.

We may also close the contour of the integral in (2.28) by adding to it the semicircle in the lower half-plane in the clockwise direction. The only singularity enclosed by this contour is the one at $z = -i$. Thus we have

$$I = -2\pi i \ \mathrm{Res}(-i) = -2\pi i \frac{1}{-2i} = \pi,$$

which is the same answer. Note that the minus sign above is due to the fact that the closed contour is clockwise.

One of the first things we do in applying the Cauchy residue theorem is to make sure that the contour is a closed one. If the contour is not closed, try to close it if possible. The second step is to locate the singularities of the integrand enclosed by the contour, and calculate the residues of the integrand at each of the singularities.

Many more examples are given in the textbook.

Homework Problems

1. Prove that the limit of (2.5) is the same for any Δz if the Cauchy-Riemann equations are

satisfied by the real part and the imaginary part of $f(z)$.

2. Prove that

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}.d.\int_0^1 \frac{1}{1+x^2}\sqrt{\frac{x^3}{1-x}}.$$

Ans. $\pi - \dfrac{\pi\cos(\pi/8)}{2^{1/4}}$.

Hint: Let $e^{\pi i/5} \equiv c + is,$ then

$$(c+is)^5 = -1.$$

Equate the imaginary parts of the two sides of this equation.

3. Evaluate the following integrals with contour integration:

a.

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x-2i)(x-3i)(x-4i)}.$$

Ans. $-\dfrac{i\pi}{60}$.

b.

$$\int_{-\infty}^{\infty} \frac{1-\cos 2x}{x^2}dx.$$

Ans. 2π.

c.

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}dx.$$

Ans. $3\pi/4$.

d.

$$\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}d\theta\ (a > b > 0).$$

Ans. $2\pi a/(a^2-b^2)^{3/2}$.

e.

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2-4\pi^2}dx.$$

Ans. π.

4. Explain why the integral of (2.35) is not equal to the imaginary part of $\int_c \frac{e^{iz}}{z}dz$.

5. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x+i}dx.$$

Explain why it is not fruitful to evaluate the integral

$$J = \int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}dx.$$

Ans. $\frac{\pi}{e}$.

6. Consider $f(z) = \log\dfrac{1+\sqrt{1+z^2}}{2}$.

a. Find all possible branch points of this function.
Ans. $0, \pm i, \infty$.

b. If we define $\sqrt{1+z^2}\,|_{z=0} = 1,$

show that the origin is not a branch point of this function. Draw a set of branch cuts to make the function single-valued.

7. Show that the Taylor series (2.20) is convergent inside the circle with center at z_0 and with the radius equal to $|\ z_0 - z_1\ |$, where z_1 is the singularity of $f(z)$ nearest to z_0.

Hint: Estimate the magnitude of $f^n(z_0)$ with the use of the Cauchy integral formula.

8. Let $f(z)$ and $g(z)$ be analytic in a region R, and let z_0 be an interior point of R. If $f(z) = g(z)$ has at least one root in any neighborhood of z_0, no matter how small this neighborhood is, prove that $f(z) = g(z)$ in R.

Hint: Let $G(z) \equiv f(z) - g(z)$ and consider the Taylor series expansion of $G(z)$ around z_0. Show that, unless this series vanishes identically, it cannot vanish at z if z is sufficiently close to z_0 but not equal to z_0.

9. Let $I_n = \int_0^\infty \frac{dx}{1+x^n}$.

a. Prove that $I_n = \frac{\pi}{n\sin(\pi/n)}$. What is I_n in the limit $n \to \infty$?

b. Show that, as $n \to \infty$, the limit of the integral I_n is equal to the limit of the integrand $(1+x^n)^{-1}$ integrated over $[0,\infty]$.

10. Evaluate the following integrals making use of branch cuts:

a.

$$\int_0^\infty \frac{\ln^2 x}{1+x^2}\,dx.$$

Ans. $\pi^3/8$.

b.

$$\int_0^1 \frac{1}{1+x^2}\sqrt{\frac{x^3}{1-x}}\,dx.$$

Ans. $\pi - \frac{\pi\cos(\pi/8)}{2^{1/4}}$.

11. Let z_0 be an isolated singularity of $f(z)$, and let z be a point in the neighborhood of z_0. Show that

$$f(z) = \int_{C_R} \frac{f(z')}{z'-z}dz' - \int_{C_\epsilon} \frac{f(z')}{z'-z}dz',$$

where C_R and C_ϵ are counterclockwise circles the centers of which are at z_0 and the radii of which are R and ϵ, respectively. Also, R is sufficiently large so that z is inside C_R, and ϵ is sufficiently small so that z is outside C_ϵ . Derive the Laurent series expansion of $f(z)$ from the equation above and discuss the region where the series is convergent.

12. Find the Fourier coefficient a_n for the following functions. What is the value of the Fourier series at $\theta = \pi$?

a. e^θ.

Ans. $a_n = \frac{(-1)^n}{2\pi}\frac{e^\pi - e^{-\pi}}{1-in}$. The value of the series at $\theta = \pi$ is $\frac{1}{2}(e^\pi + e^{-\pi})$.

b. $\frac{1}{a + b\cos\theta}$.

Ans. $a_n = 2\pi(-1)^n\left(a - \sqrt{a^2-b^2}\right)^n/\left(b^n\sqrt{a^2-b^2}\right)$ $(n > 0)$, $a_{-n} = a_n$. The value of the Fourier series at $\theta = \pi$ is $(a-b)^{-1}$.

13. Find the Fourier transform of the following functions:

a. $e^{-|x|}$.

Ans. $\frac{2}{1+k^2.}$.

b. $(1+x^2)^{-2}$.

Ans. $\frac{\pi(1+|\ k\ |)}{2}e^{-|k|}$.