18.305 Exam 1,

October 18, 04. Closed Book.

Problem: Consider the differential equation

$y" + x^4 y = 0.$

(a) Locate and classify the singular points, finite or infinite, of this differential equation. (10%)

(b) Find the WKB solutions of this equation. For what values of x are the WKB solutions good approximations of the solution? (25%)

(c) Find the Maclaurin series solution of this equation. For what values of x are these series convergent? (25%)

(d) Find the entire asymptotic series of the solutions which are useful when the magnitude of x is very large. For what values are these series convergent? (25%)

(e) Find the exact solution of this equatiion. (15%)

Solutions:

(a) and (b):

The infinity is an irregular singular point of this equation.

Since x^4 is always positive, the WKB solutions are oscillatory. Putting

$p = x^2, \quad \int p dx = x^3/3,$

we have

$y^{\pm}_{WKB} = x^{-1} \exp(\pm i x^3/3).$

The rank of the irregular singular point at ∞ is 3.

c. Let

$y = \sum a_n x^n, \quad a_{-1} = a_{-2} = \bullet\bullet\bullet = 0.$

We have

$y" = \sum a_n n(n-1) x^{n-2}$

and

$x^4 y = \sum a_n x^{n+4} = \sum a_{n-6} x^{n-2}.$

Thus the recurrence formula is

$a_n n(n-1) = -a_{n-6}.$

Let

$n = 6m,$

then

$$a_{6m} = -\frac{a_{6(m-1)}}{6^2 m(m-1/6)} = (-1)^m \frac{a_0 \Gamma(5/6)}{6^{2m} m! \Gamma(m+5/6)}.$$

Thus one of the Maclaurin solutions is

$$y_1 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{6m}}{6^{2m} m! \Gamma(m+5/6)}.$$

Setting

$n = 6m+1,$

we have

$$a_{6m+1} = -\frac{a_{6(m-1)+1}}{6^2 (m+1/6)(m)} = (-1)^m \frac{a_1 \Gamma(7/6)}{6^{2m} m! \Gamma(m+7/6)}.$$

Thus the second Maclaurin solution is

$$y_2 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{6m+1}}{6^{2m} m! \Gamma(m+7/6)}.$$

These seires converge for all finite values of x.

(d) Let

$$y = \exp(ix^3/3)Y,$$

then

$$(D + ix^2)(D + ix^2)Y + x^4 Y = 0,$$

or

$$Y'' + 2ix^2 Y' + 2ixY = 0.$$

Let

$$Y = \sum A_n x^{-1-n}, \; A_{-1} = A_{-2} = \bullet\bullet\bullet = 0.$$

We have

$$Y'' = \sum A_n (n+1)(n+2)x^{-3-n} = \sum A_{n-3}(n-2)(n-1)x^{-n}$$

and

$$2ix^2 Y' + 2ixY = -2i \sum A_n n x^{-n}.$$

We get

$$A_n = \frac{(n-2)(n-1)}{2in} A_{n-3}, \; n > 0.$$

Setting $n = 3m$, we get

$$A_{3m} = \frac{3(m-2/3)(m-1/3)}{2im} A_{3(m-1)} = (\frac{3}{2i})^m \frac{\Gamma(m+1/3)\Gamma(m+2/3)}{m!\Gamma(2/3)\Gamma(1/3)}.$$

Thus one of the asymptotic solutions is

$$y_3 = e^{ix^3/3} \sum_{m=0}^{\infty} \frac{(3/2i)^m \Gamma(m+1/3)\Gamma(m+2/3)x^{-3m-1}}{m!}.$$

The second asymptotic solution is obtained by taking the complex conjugate of y_3. We get

$$y_4 = e^{-ix^3/3} \sum_{m=0}^{\infty} \frac{(3i/2)^m \Gamma(m+1/3)\Gamma(m+2/3)x^{-3m-1}}{m!}$$

These series converge for no value of x.

(e) The asymptotic forms of the Bessel functions are $t^{-1/2}\exp(\pm it)$.

Thus we put

$$x^3/3 = t, \quad y = t^{1/6} Z.$$

We have

$$\frac{d}{dx} = 3^{2/3} t^{2/3} \frac{d}{dt}.$$

Therefore

$$\frac{d^2 y}{dx^2} = 3^{4/3} t^{2/3} \frac{d}{dt} t^{2/3} \frac{d}{dt} t^{1/6} Z = 3^{4/3} t^{3/2} (\frac{d}{dt} + \frac{5}{6t})(\frac{d}{dt} + \frac{1}{6t})Z,$$

and the differential equation for Z is

$$\frac{d^2 Z}{dt^2} + \frac{1}{t}\frac{dZ}{dt} + (1 - \frac{1}{36t^2})Z = 0.$$

The solution of the equation above is,

$$Z = aJ_{1/6}(t) + bJ_{-1/6}(t).$$

Hence

$$y(x) = ax^{1/2} J_{1/6}(\frac{x^3}{3}) + bx^{1/2} J_{-1/6}(\frac{x^3}{3}).$$