18.305 Fall 2004/05
Solutions to Assignment 4: The Laplace method
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1. **Find the leading term for each of the integrals below for $\lambda >> 1$.**
   (a) $\int_{-1}^{4} e^{-\lambda x^3}(1+x^4)dx$
   (b) $\int_{1}^{\infty} \sqrt{x-1}e^{-\lambda \cosh x}dx$
   (c) $\int_{0}^{2} e^{\lambda x(1-x)}dx$

2. **Find the leading term for each of the integrals below $\lambda >> 1$.**
   (a) $\int_{-1}^{1} e^{-\lambda x^3}dx$
   (b) $\int_{1}^{\infty} e^{-\lambda x^2}dx$
   (c) $\int_{-2}^{1}(\sin x)e^{-\lambda x^2}dx$
   (d) $\int_{-\pi}^{\pi} e^{-\lambda \sin x}dx$
   (e) $\int_{0}^{\infty} e^{-\lambda x}e^{-x^2}dx$
   (f) $\int_{0}^{\lambda} e^{x^3}dx$
   (g) $\int_{0}^{\infty} e^{-\lambda(x+x^5)}dx$

3. **Find the entire asymptotic series for each of the integrals in problem 2.**

Solutions:
In the following, we assume the given integrals to be in the form

$$I(\lambda) = \int_a^b h(x)e^{-\lambda v(x)}dx$$

1. **(a)** $v(x) = x^3$, which has a minimum at the lower end point -1. Since $v(x)$ is monotonically increasing in $[-1, 1]$, we can use the formula

$$I(\lambda) \approx \frac{e^{-\lambda v(a)}h(a)}{\lambda v'(a)} \qquad (1)$$

to obtain the leading term as

$$I(\lambda) \approx \frac{2e^{\lambda}}{3\lambda}$$

(b) The integral can be written as

$$\begin{aligned} I(\lambda) &= \int_0^\infty t^{1/2} e^{-\lambda \cosh(t+1)} dt \\ &= \int_0^\infty t^{1/2} e^{-\lambda[\cosh 1 + t \sinh 1 + ..]} dt \\ &\approx e^{-\lambda \cosh 1} \int_0^\infty t^{1/2} e^{-\lambda t \sinh 1} dt \\ &= e^{-\lambda \cosh 1} \int_0^\infty (\frac{1}{\lambda \sinh 1})^{3/2} s^{1/2} e^{-s} ds \\ &= \frac{e^{-\lambda \cosh 1}}{(\lambda \sinh 1)^{3/2}} \Gamma(3/2) = \frac{e^{-\lambda \cosh 1}}{(\lambda \sinh 1)^{3/2}} \frac{\sqrt{\pi}}{2} \end{aligned}$$

(c) $v(x) = -x(1-x)$ takes its minimum at $x = 1/2$, which is an interior point. As $v''(1/2) \neq 0$, we can use the formula

$$I(\lambda) \approx \sqrt{\frac{2\pi}{\lambda |v''(x_0)|}} e^{-\lambda v(x_0)} h(x_0) \tag{2}$$

to obtain the leading term

$$I(\lambda) \approx \sqrt{\frac{\pi}{\lambda}} e^{\lambda/4}$$

2. **(a)** $v(x) = x^3$, which takes its minimum at $x = -1$. So, by using (1), we find

$$I(\lambda) \approx \frac{e^\lambda}{3\lambda}$$

(b) $v(x) = x^2$, takes its minimum at $x = 1$. Therefore, using (1), we find

$$I(\lambda) \approx \frac{e^{-\lambda}}{2\lambda}$$

(c)

$$\int_{-2}^{1} (\sin x) e^{-\lambda x^2} dx = \int_{-2}^{-1} (\sin x) e^{-\lambda x^2} dx + \int_{-1}^{1} (\sin x) e^{-\lambda x^2} dx$$

where the second integral is zero, because its integrand is odd. Therefore, we only consider the first integral. $v(x) = x^2$, which takes its minimum at $x = -1$ and is monotonically decreasing throughout $[-2, -1]$. Therefore, by using the formula

$$I(\lambda) \approx -\frac{e^{-\lambda v(b)} h(b)}{\lambda v'(b)} \tag{3}$$

to obtain the leading term as

$$I(\lambda) \approx -\frac{e^{-\lambda}}{2\lambda} \sin 1$$

(d) $v(x) = \sin x$ takes its minimum at $x = -\frac{\pi}{2}$, an interior point. Therefore, the formula (2) gives the leading term

$$\sqrt{\frac{2\pi}{\lambda}} e^{\lambda}$$

(e) $h(x) = e^{-x^2}$ and $v(x) = x$, which takes its minimum at $x = 0$ and is monotonic throughout the domain of integration. Therefore, the relevant formula is (3), which gives

$$I(\lambda) \approx \frac{1}{\lambda}$$

(f) Since the main contribution comes from $x = \lambda$ part, we can replace the integral by

$$\begin{aligned} I(\lambda) &= \int_1^{\lambda} e^{x^3} dx \\ &= \int_1^{\lambda} \frac{1}{3x^2} 3x^2 e^{x^3} dx = \frac{1}{3x^2} e^{x^3}\Big|_1^{\lambda} + \int_1^{\lambda} \frac{2}{3x^3} e^{x^3} dx \end{aligned}$$

which implies the leading term

$$\frac{1}{3\lambda^2} e^{\lambda^3}$$

(g) $v(x) = x + x^5$, which takes on its minimum at $x = 0$, therefore by using the formula (3), we obtain

$$\frac{1}{\lambda}$$

3. **(a)** We first let $s = x^3 + 1$, then the integral becomes

$$I(\lambda) = e^{\lambda} \int_0^2 e^{-\lambda s} \frac{1}{3} (1-s)^{2/3} dx$$

where now the contribution comes from $s = 0$. So we can change the upper limit to ∞. We further let $\rho = \lambda s$, to obtain

$$I(\lambda) = \frac{e^{\lambda}}{3\lambda} \int_0^{\infty} e^{-\rho} (-\frac{\rho}{\lambda} + 1)^{2/3} dx$$

The idea behind all those transformations is to have the leading term $\frac{e^{\lambda}}{3\lambda}$ outside the integral, as above. Now we expand, and get

$$I(\lambda) = \frac{e^{\lambda}}{3\lambda} \int_0^{\infty} e^{-\rho} \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \frac{\Gamma(2/3)}{\Gamma(2/3 - k)} (\frac{\rho}{\lambda})^k dx = \frac{e^{\lambda}}{3\lambda} \int_0^{\infty} e^{-\rho} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(2/3 + k)}{\Gamma(2/3)} (\frac{\rho}{\lambda})^k dx$$

We illegitimately change the order of integration and summation, to obtain the asymptotic series

$$\begin{aligned} I(\lambda) &= \frac{e^{\lambda}}{3\lambda} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(2/3+k)}{\Gamma(2/3)} \frac{1}{\lambda^k} \int_0^{\infty} e^{-\rho} \rho^k dx \\ &= \frac{e^{\lambda}}{3\lambda} \sum_{k=0}^{\infty} \frac{\Gamma(2/3+k)}{\Gamma(2/3)} \frac{1}{\lambda^k} \end{aligned}$$

(b) We first let $s = x^2 - 1$, then the integral becomes

$$I(\lambda) = e^{-\lambda} \int_0^2 e^{-\lambda s} \frac{1}{2} (s+1)^{-1/2} ds$$

where now the contribution comes from $s = 0$. So we can change the upper limit to ∞. We further let $\rho = \lambda s$, to obtain

$$I(\lambda) = \frac{e^{-\lambda}}{2\lambda} \int_0^\infty e^{-\rho} (1 + \frac{\rho}{\lambda})^{-1/2} dx$$

The idea behind all those transformations is to have the leading term $\frac{e^{\lambda}}{2\lambda}$ outside the integral, as above. Now we expand, and get

$$I(\lambda) = \frac{e^{-\lambda}}{2\lambda} \int_0^\infty e^{-\rho} \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \frac{\Gamma(1/2 + k)}{\Gamma(1/2)} (\frac{\rho}{\lambda})^k dx$$

We illegitimately change the order of integration and summation, to obtain the asymptotic series

$$\begin{aligned} I(\lambda) &= \frac{e^{-\lambda}}{2\lambda} \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \frac{\Gamma(1/2 + k)}{\Gamma(1/2)} \frac{1}{\lambda^k} \int_0^\infty e^{-\rho} \rho^k dx \\ &= \frac{e^{-\lambda}}{2\lambda} \sum_{k=0}^{\infty} \frac{\Gamma(1/2 + k)}{\Gamma(1/2)} (-1)^k \frac{1}{\lambda^k} \end{aligned}$$

(c) We consider only

$$I(\lambda) = \int_{-2}^{-1} (\sin x) e^{-\lambda x^2} dx$$

first let $s = x + 1$, to obtain

$$I(\lambda) = \int_{-1}^{0} \sin(s-1) e^{-\lambda[s^2 - 2s + 1]} ds \approx e^{-\lambda} \int_{-\infty}^{0} \sin(s-1) e^{-\lambda[s^2 - 2s]} ds$$

then we further let $\rho = -2\lambda s$, to obtain

$$I(\lambda) \approx e^{-\lambda} \int_0^\infty \sin(1 + (\frac{\rho}{2\lambda})) e^{-\rho} e^{-\rho^2/4\lambda} \frac{d\rho}{2\lambda}$$

Then, expanding

$$\sin(1 + (\frac{\rho}{2\lambda})) = \sin 1 + \frac{1}{1!} \cos 1 (\frac{\rho}{2\lambda}) - \frac{1}{2!} \sin 1 (\frac{\rho}{2\lambda})^2 + ...$$

and

$$e^{-\rho^2/4\lambda} = \sum_{k=0}^{\infty} (\frac{-\rho^2}{4\lambda})^k$$

and plugging those in, one may obtain the entire asymptotic series of the given integral.

(d) We first let $s = x + \frac{\pi}{2}$, as the main contribution comes from $x = -\frac{\pi}{2}$.This gives

$$I(\lambda) \approx \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-\lambda \cos s} ds \approx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\lambda \cos s} ds = 2 \int_{0}^{\frac{\pi}{2}} e^{-\lambda \cos s} ds$$

As a second step, we let $\rho = -\lambda(\cos s - 1)$, to obtain

$$I(\lambda) \approx 2e^{\lambda} \int_0^1 e^{-\rho} (\frac{\rho}{\lambda})^{-1/2} (2 - \frac{\rho}{\lambda})^{-1/2} d\rho \approx \frac{e^{\lambda}}{(2\lambda)^{1/2}} \int_0^1 e^{-\rho} \rho^{-1/2} (1 - \frac{\rho}{2\lambda})^{-1/2} d\rho$$

Changing illegitimately, the order of integration and summation, we obtain

$$I(\lambda) \approx \sqrt{2} \frac{e^{\lambda}}{\lambda^{1/2}} \sum_{k=0}^{\infty} \frac{1}{k!} (\frac{1}{2\lambda})^k \frac{\Gamma(1/2 + k)}{\Gamma(1/2)} \int_0^1 d\rho e^{-\rho} \rho^{k-1/2}$$

The asymptotic series is obtained by replacing the upper limit of the integral by ∞, and it is

$$\sqrt{2\pi} \frac{e^{\lambda}}{\lambda^{1/2}} \sum_{k=0}^{\infty} (\frac{1}{2\lambda})^k \frac{\Gamma^2(1/2 + k)}{k! \Gamma^2(1/2)}$$

(e) Letting $\rho = \lambda x$, we get

$$\begin{aligned} I(\lambda) &= \frac{1}{\lambda} \int_0^{\infty} e^{-\rho} e^{-(\frac{\rho}{\lambda})^2} d\rho \\ &= \frac{1}{\lambda} \int_0^{\infty} e^{-\rho} \sum_{k=0}^{\infty} \frac{1}{k!} (-\frac{\rho}{\lambda})^{2k} d\rho \end{aligned}$$

and so the asymptotic series is

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{k!} (-\frac{1}{\lambda})^{2k} \int_0^{\infty} d\rho e^{-\rho} \rho^{2k} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{(2k)!}{k!} (-\frac{1}{\lambda})^{2k}$$

(f) We first let $s = x^3 - \lambda^3$, to obtain

$$\begin{aligned} I(\lambda) &= e^{\lambda^3} \int_{-\lambda}^{0} e^s \frac{1}{3} (s + \lambda^3)^{-2/3} ds \\ &= \frac{e^{\lambda^3}}{3\lambda^2} \int_{-\lambda}^{0} e^s (1 + \frac{s}{\lambda^3})^{-2/3} ds \\ &= \frac{e^{\lambda^3}}{3\lambda^2} \int_{-\lambda}^{0} e^s \sum_{k=0}^{\infty} \frac{1}{k!} (\frac{s}{\lambda^3})^k (-1)^k \frac{\Gamma(2/3 + k)}{\Gamma(2/3)} ds \end{aligned}$$

Therefore the asymptotic series is

$$\begin{aligned} & \frac{e^{\lambda^3}}{3\lambda^2} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(2/3 + k)}{\Gamma(2/3)} \frac{1}{\lambda^{3k}} (-1)^k \int_{-\infty}^{0} s^k e^s ds \\ = & \frac{e^{\lambda^3}}{3\lambda^2} \sum_{k=0}^{\infty} \frac{\Gamma(2/3 + k)}{\Gamma(2/3)} \frac{1}{\lambda^{3k}} \end{aligned}$$

(g) We let $s = \lambda x$, to obtain

$$\begin{aligned} I(\lambda) &= \frac{1}{\lambda}\int_0^\infty e^{-s} e^{-s^5/\lambda^4} ds \\ &= \frac{1}{\lambda}\int_0^\infty e^{-s} \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k \frac{s^{5k}}{\lambda^{4k}} ds \end{aligned}$$

Therefore the asymptotic series is

$$\frac{1}{\lambda}\sum_{k=0}^{\infty} \frac{1}{\lambda^{4k}}\frac{1}{k!}(-1)^k \int_0^\infty e^{-s} s^{5k} ds = \frac{1}{\lambda}\sum_{k=0}^{\infty} \frac{(5k)!}{k!}(-1)^k \frac{1}{\lambda^{4k}}$$