18.305 Fall 2004/05

Solutions to Assignment 2: Asymptotic Series and WKB

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1. **(Chapter 6, Problem 8)Find the entire asymptotic series for the solutions of the following ODE:**

   (a) $xy'' + (c - x)y' - ay = 0$ **(confluent hypergeometric equation)**

   (b) $x(1 - x)y'' + [c - (a + b + 1)x]y' - aby = 0.$ **(hypergeometric equation)**

   (c) $y'' - (x^4 - \frac{3}{16}x^{-2})y = 0.$

   (d) $y'' + (x^2 + \frac{3}{16}x^{-2})y = 0.$

   (e) $y'' + (\nu + \frac{1}{2} - \frac{1}{4}x^2)y = 0$, ν **a constant. (parabolic cylinder equation)**

Solution:

Prelimenaries: pages 172-177 in the book.

(a) Since $x = \infty$ is an irregular singular point of rank 1, we first make the change of variables

$$y = e^{Ax}Y$$

which leads to $D \to D + A$, and the ODE becomes

$$[(D + A)^2 + (\frac{c}{x} - 1)(D + A) - \frac{a}{x}]Y = 0$$

or

$$\{D^2 + [2A + \frac{c}{x} - 1]D + [A(\frac{c}{x} - 1) + A^2 - \frac{a}{x}]\}Y = 0 \tag{1}$$

With $t = \frac{1}{x}$, we need to find A such that the term $\frac{1}{t^4}d(\frac{1}{t})$ (here d refers to the coefficient of y in the original ODE, to see where this comes from, refer to page 177 in the book) does not have a pole of order higher that $(k + 2) = 3$. So we need to have the function $d(\frac{1}{t}) = Act - A + A^2 - at$ to have a factor of t, which is possible if $-A + A^2 = 0$. Then either $A = 0$ or $A = 1$. Each case will be treated separately. First, let us take $A = 0$. We plug

$$Y = \sum_{n=-\infty}^{\infty} a_n x^{-n-s} \tag{2}$$

(in this case $y = Y$) into the ODE to obtain

$$\sum_{n=-\infty}^{\infty} [(n + s)(n + s + 1) - c(n + s)]a_n x^{-n-s-1} + \sum_{n=-\infty}^{\infty} [(n + s) - a]a_n x^{-n-s} = 0$$

which leads to, after making $n \to n - 1$ in the first summation,

$$(n + s - a)a_n + (n + s - 1)(n + s - c)a_{n-1} = 0$$

Letting $n = 0$, we find that $s = a$. Rewriting,

$$a_n = -\frac{(n + s - 1)(n + s - c)}{(n + s - a)}a_{n-1}$$

Thus

$$a_n = (-1)^n \frac{\Gamma(n+a)\Gamma(n+1+a-c)}{n!\Gamma(a)\Gamma(a-c+1)}$$

one solution is

$$\boxed{y_1(x) = x^{-a} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+a)\Gamma(n+1+a-c)}{n!} x^{-n}}$$

To find the other solution, we let $A = 1$ in (1), and obtain

$$[D^2 + (1 + \frac{c}{x})D + \frac{c-a}{x}]Y = 0$$

Again, we plug in (2) into this last equation, to obtain

$$\sum_{n=-\infty}^{\infty} [(n+s)(n+s+1) - c(n+s)]a_n x^{-n-s-2} + \sum_{n=-\infty}^{\infty} [-(n+s) + c - a]a_n x^{-n-s-1} = 0$$

which gives us, with $n \to n-1$ in the first summation, that

$$(n-1+s)(n+s-c)a_{n-1} - (n+s-c+a)a_n = 0$$

Letting $n = 0$, we find that $s = c - a$. Putting this into the last formula, we have

$$a_n = \frac{(n-1+c-a)(n-a)}{n} a_{n-1}$$

which gives

$$a_n = \frac{\Gamma(n+c-a)\Gamma(n+1-a)}{n!\Gamma(c-a)\Gamma(1-a)} a_{n-1}$$

Thus, the second solution is

$$\boxed{y_2(x) = x^{a-c} e^x \sum_{n=0}^{\infty} \frac{\Gamma(n+c-a)\Gamma(n+1-a)}{n!} x^{-n}}$$

The general solution is

$$\boxed{y(x) = C_1 x^{-a} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+a)\Gamma(n+1+a-c)}{n!} x^{-n} + C_2 x^{a-c} e^x \sum_{n=0}^{\infty} \frac{\Gamma(n+c-a)\Gamma(n+1-a)}{n!} x^{-n}}$$

where C_1 and C_2 are arbitrary constants. Note that series for both of the solutions are asymptotic series, they converge nowhere.

(b) Since $x = \infty$ is a regular singular point, we directly plug in the series (2) into the ODE in question to obtain

$$\sum_{n=-\infty}^{\infty} (n+s)(n+s+1-c)a_n x^{-n-s-1} + \sum_{n=-\infty}^{\infty} [-(n+s)(n+s+1) + (a+b+1)(n+s) - ab]a_n x^{-n-s} = 0$$

After making $n \to n-1$ in the first summation, we obtain

$$(n+s-a)(n+s-b)a_n = (n-1+s)(n+s-c)a_{n-1}$$

Letting $n=0$, we find that either $s=a$, or $s=b$. Rewriting the above formula

$$a_n = \frac{(n-1+s)(n+s-c)}{(n+s-a)(n+s-b)} a_{n-1}$$

which gives

$$a_n = \frac{\Gamma(n+s)\Gamma(n+1+s-c)}{\Gamma(n+1+s-a)\Gamma(n+1+s-b)} \frac{\Gamma(1+s-a)\Gamma(1+s-b)}{\Gamma(s)\Gamma(1+s-c)} a_0$$

Thus the general solution is

$$\boxed{y = C_1 x^{-a} \sum_{n=0}^{\infty} \frac{\Gamma(n+a)\Gamma(n+1+a-c)}{n!\Gamma(n+1+a-b)} x^{-n} + C_2 x^{-b} \sum_{n=0}^{\infty} \frac{\Gamma(n+b)\Gamma(n+1+b-c)}{n!\Gamma(n+1+b-a)} x^{-n}}$$

where C_1 and C_2 are arbitrary constants. Note that those series are convergent for $|x|>1$.

(c) Since $x=\infty$ is an irregular singular point of rank 3, we seek a coordinate transformation of the form

$$y = e^{A_1 x^3 + A_2 x^2 + A_3 x} Y$$

which will transform the ODE into a form in which the coeefficent of Y does not have a pole of order higher than $(k+2)=5$. In other words, after transformation the quantity $\frac{1}{t^4} d(\frac{1}{t})$ should not have a pole of order higher than 5, which means $d(\frac{1}{t})$ should not have any poles of order higher than 1 at $t=0$. The ODE for Y is

$$[(D + 3x^2 A_1 + 2x A_2 + A_3)^2 - x^4 + \frac{3}{16} x^{-2}]Y = 0$$

$\Rightarrow$

$$\{D^2 + [6x^2 A_1 + 4x A_2 + 2A_3]D + [(9A_1^2 - 1)x^4 + 12 A_1 A_2 x^3 + (3A_1 A_3 + 4A_2^2)x^2 + (4A_2 A_3 + 6A_1)x + (A_3^2 + 2A_2) +$$

We see that we need to do is to eliminate the x^4, x^3, x^2 terms in the coeefficient of Y. This can easily be done by letting $9A_1^2 - 1 = 0$, $A_2 = A_3 = 0$. Hence there are two cases, namely $A_1 = \pm\frac{1}{3}$.

Let's first analyze $A_1 = \frac{1}{3}$. Then indeed the ODE becomes

$$[D^2 + 2x^2 D + 2x + \frac{3}{16} x^{-2}]Y = 0$$

which has $d(\frac{1}{t}) = 2\frac{1}{t} + \frac{3}{16} t^2$. This last quantity has no poles of order higher than 1, verifying our earlier remark.

Now we proceed as in the earlier cases, thus we plug (2) into the ODE. This gives us

$$\sum_{n=-\infty}^{\infty} \underbrace{[(n+s)(n+s+1) + \frac{3}{16}]}_{(n+s+\frac{1}{4})(n+s+\frac{3}{4})} a_n x^{-n-s-2} + \sum_{n=-\infty}^{\infty} [-2(n+s)+2] a_n x^{-n-s+1} = 0$$

Making $n \to n-3$ in the first summation, we obtain

$$2(n+s-1)a_n + (n-3+s+\frac{1}{4})(n-3+s+\frac{3}{4})a_{n-3} = 0$$

Letting $n = 0$ gives that $s = 1$. Using this, we rewrite the above formula with $n = 3m$, we obtain

$$a_{3m} = \frac{3(m-\frac{7}{12})(m-\frac{5}{12})}{2m} a_{3(m-1)}$$

Thus the one of the solutions which is valid for $|x| >> 1$ is

$$\boxed{y_1(x)=x^{-1}e^{\frac{1}{3}x^3}\sum_{n=0}^{\infty}(\tfrac{3}{2})^n\frac{\Gamma(n+\frac{5}{12})\Gamma(n+\frac{7}{12})}{n!}x^{-3n}}$$

For the case when we take $A_1 = -\frac{1}{3}$, our ODE is

$$[D^2 - 2x^2D - 2x + \frac{3}{16}x^{-2}]Y = 0$$

Plugging in (2) into the ODE. This gives us

$$\sum_{n=-\infty}^{\infty} \underbrace{[(n+s)(n+s+1)+\frac{3}{16}]}_{(n+s+\frac{1}{4})(n+s+\frac{3}{4})} a_n x^{-n-s-2} + \sum_{n=-\infty}^{\infty}[2(n+s)-2]a_n x^{-n-s+1} = 0$$

Making $n \to n-3$ in the first summation, we obtain

$$2(n+s-1)a_n + (n-3+s+\frac{1}{4})(n-3+s+\frac{3}{4})a_{n-3} = 0$$

Letting $n = 0$ gives that $s = 1$. Using this and rewriting the above formula with $n = 3m$, we obtain

$$a_{3m} = -\frac{3(m-\frac{7}{12})(m-\frac{5}{12})}{2m} a_{3(m-1)}$$

Thus the second solution which is valid for $|x| >> 1$ is

$$\boxed{y_2(x)=x^{-1}e^{-\frac{1}{3}x^3}\sum_{n=0}^{\infty}(-\tfrac{3}{2})^n\frac{\Gamma(n+\frac{5}{12})\Gamma(n+\frac{7}{12})}{n!}x^{-3n}}$$

The general solution is a linear combination of those two solutions. Note that both series are convergent nowhere.

(d) Since $x = \infty$ is an irregular singular point of rank 2, we seek a coordinate transformation of the form

$$y = e^{A_1x^2+A_2x}Y$$

which will transform the ODE into a form in which the coeefficent of Y does not have a pole of order higher than $(k+2) = 4$. In other words, after transformation the quantity $\frac{1}{t^4}d(\frac{1}{t})$ should not have a pole of order higher than 4, which means $d(\frac{1}{t})$ should not have any poles at $t = 0$. The ODE for Y is

$$[(D + 2xA_1 + A_2)^2 + x^2 + \frac{3}{16}x^{-2}]Y = 0$$

or

$$[D^2 + 4xA_1D + 2A_1 + 4x^2A_1^2 + 4xA_1A_2 + A_2^2 + x^2 + \frac{3}{16}x^{-2}]Y = 0$$

We need to do is to eliminate the x^2 and x terms in the coefficient of Y. This can easily be done by letting $4A_1^2 + 1 = 0$, $A_2 = 0$. Let's choose $A_1 = \pm\frac{1}{2}i, A_2 = 0$. Then indeed the ODE becomes

$$[D^2 \pm 2xiD \pm i + \frac{3}{16}x^{-2}]Y = 0$$

which has $d(\frac{1}{t}) = \pm i + \frac{3}{16}t^2$. This last quantity has no poles, as we wished.

Now we proceed as in the earlier cases, thus we plug (2) into the ODE. This gives us

$$\sum_{n=-\infty}^{\infty}[(n+s)(n+s+1) + \frac{3}{16}]a_n x^{-n-s-2} \pm i \sum_{n=-\infty}^{\infty}[-2(n+s)+1]a_n x^{-n-s} = 0$$

Making $n \to n-2$ in the first summation, we obtain

$$\pm ia_n(-2(n+s)+1) + (n-2+s+\frac{1}{4})(n-2+s+\frac{3}{4})a_{n-2} = 0$$

Letting $n = 0$ gives that $s = \frac{1}{2}$ (in both cases). Rewriting the above formula, we have

$$a_n = \pm i\frac{(n-\frac{5}{4})(n-\frac{3}{4})}{2n}a_{n-2}$$

With $n = 2m$, the above formula is

$$a_{2m} = \pm i\frac{(m-\frac{5}{8})(m-\frac{3}{8})}{m}a_{2(m-1)}$$

Thus the general solution which is valid for $|x| >> 1$ is

$$y = C_1x^{-\frac{1}{2}}e^{\frac{1}{2}ix^2}\sum_{n=0}^{\infty}(i)^{-n}\frac{\Gamma(n+\frac{3}{8})\Gamma(n+\frac{5}{8})}{n!}x^{-2n} + C_2x^{-\frac{1}{2}}e^{-\frac{1}{2}ix^2}\sum_{n=0}^{\infty}(i)^{n}\frac{\Gamma(n+\frac{3}{8})\Gamma(n+\frac{5}{8})}{n!}x^{-2n}$$

where C_1 and C_2 are arbitrary constants. The second solution can also be obtained by taking the complex conjugate of the first solution. Note that both series converge nowhere.

(e) Since $x = \infty$ is an irregular singular point of rank 2, we seek a coordinate transformation of the form

$$y = e^{A_1x^2 + A_2x}Y$$

which will transform the ODE into a form in which the coeefficent of Y does not have a pole of order higher than $(k+2) = 4$. In other words, after transformation the quantity $\frac{1}{t^4}d(\frac{1}{t})$ should not have a pole of order higher than 4, which means $d(\frac{1}{t})$ should not have any poles at $t = 0$. The ODE for Y is

$$[(D + 2xA_1 + A_2)^2 + \nu + \frac{1}{2} - \frac{1}{4}x^2]Y = 0$$

or

$$[D^2 + 4xA_1D + 2A_1 + 4x^2A_1^2 + 4xA_1A_2 + A_2^2 - \frac{1}{4}x^2]Y = 0$$

We see that the only thing we need to do is to eliminate the x^2 and x terms in the coefficient of Y. This can easily be done by letting $4A_1^2 - \frac{1}{4} = 0$, $A_2 = 0$.

Let's proceed with $A_1 = \pm\frac{1}{4}$. Then indeed the ODE becomes

$$[D^2 \pm xD + \nu + \frac{1}{2} \pm \frac{1}{2}]Y = 0$$

which has $d(\frac{1}{t}) = \nu + \frac{1}{2} \pm \frac{1}{2}$. This last quantity has no poles, as we wished.

Now we proceed as in the earlier cases, thus we plug (2) into the ODE. This gives us

$$\sum_{n=-\infty}^{\infty}(n+s)(n+s+1)a_nx^{-n-s-2} + \sum_{n=-\infty}^{\infty}[\mp(n+s) + \nu + \frac{1}{2} \pm \frac{1}{2}]a_nx^{-n-s} = 0$$

Making $n \to n-2$ in the first summation, we obtain

$$(\mp(n+s) + \nu + \frac{1}{2} \pm \frac{1}{2})a_n + (n-2+s)(n-1+s)a_{n-2} = 0$$

Letting $n = 0$, we find that $s = \pm\nu + \frac{1}{2} \pm \frac{1}{2}$. Rewriting the above formula

$$a_n = -\frac{(n-2+s)(n-1+s)}{\mp(n+s) + \nu + \frac{1}{2} \pm \frac{1}{2}}a_{n-2}$$

With $n = 2m$, the above formula is

$$a_{2m} = -\frac{(2m-2+s)(2m-1+s)}{\mp(2m+s) + \nu + \frac{1}{2} \pm \frac{1}{2}}a_{2(m-1)} = -2\frac{(m+\frac{s-2}{2})(m+\frac{s-1}{2})}{[\mp m + \frac{1}{2}(\mp s + \nu + \frac{1}{2} \pm \frac{1}{2})]}a_{2(m-1)}$$

Hence for upper (+) case: $s = \nu + 1$, and

$$a_{2m} = 2\frac{(m+\frac{\nu-1}{2})(m+\frac{\nu}{2})}{m}a_{2(m-1)} = 2^m\frac{\Gamma(m+\frac{\nu+1}{2})\Gamma(m+\frac{\nu+2}{2})}{m!\Gamma(\frac{\nu-1}{2})\Gamma(\frac{\nu}{2})}a_0$$

and for the lower (−) case: $s = -\nu$, and

$$a_{2m} = -2\frac{(m-\frac{\nu+2}{2})(m-\frac{\nu+1}{2})}{m+\frac{1}{2}}a_{2(m-1)} = (-2)^m\frac{\Gamma(m-\frac{\nu}{2})\Gamma(m-\frac{\nu-1}{2})\Gamma(\frac{1}{2})}{\Gamma(m+\frac{3}{2})\Gamma(-\frac{\nu}{2})\Gamma(\frac{-\nu+1}{2})}a_0$$

Thus the general solution which is valid for $|x| >> 1$ is

$$y = C_1e^{\frac{1}{4}x^2}x^{-\nu-1}\sum_{n=0}^{\infty}2^n\frac{\Gamma(n+\frac{\nu+1}{2})\Gamma(n+\frac{\nu+2}{2})}{n!}x^{-2n} + C_2e^{-\frac{1}{4}x^2}x^{\nu}\sum_{n=0}^{\infty}(-2)^n\frac{\Gamma(n-\frac{\nu}{2})\Gamma(n-\frac{\nu-1}{2})}{n!}x^{-2n}$$

where C_1 and C_2 are arbitrary constants. Note that both series are asymptotic series, they converge nowhere.

2. (Chapter 7,Problem 1) Show that the Wronskian of y^+_{WKB} and y^-_{WKB} given by (7.5) is a constant.

Solution:

The Wronskian is defined to be

$$W = y_1 y_2' - y_1' y_2$$

where, in our case,

$$\begin{aligned} y_1 &= y^+_{WKB} = \frac{1}{\sqrt{p(x)}} e^{i\int p(x)dx} \\ y_2 &= y^-_{WKB} = \frac{1}{\sqrt{p(x)}} e^{-i\int p(x)dx} \end{aligned} \tag{3}$$

We differentiate those to find

$$(y^{\pm}_{WKB})' = [\pm ip(x) - \frac{1}{2p(x)}] y^{\pm}_{WKB}$$

Thus

$$\begin{aligned} W &= y^+_{WKB} y^-_{WKB} [-ip(x) - \frac{1}{2p(x)}] - [ip(x) - \frac{1}{2p(x)}] y^+_{WKB} y^-_{WKB} \\ &= -2ip(x) y^+_{WKB} y^-_{WKB} \\ &= -2ip(x) \frac{1}{\sqrt{p(x)}} e^{i\int p(x)dx} \frac{1}{\sqrt{p(x)}} e^{-i\int p(x)dx} = -2i = \text{constant} \end{aligned}$$

3. The WKB solutions (3) can also be derived by putting

$$y = e^{iS}$$

(a) Substitute WKB solutions into $y'' + p^2 y = 0$ and show that

$$iS'' - (S')^2 + p^2 = 0$$

which is a nonlinear ODE.

(b) If $p(x)$ is of the form

$$p(x) = \lambda P(x)$$

give a reason which suggests that we may drop the term iS'' in the equation above and obtain

$$(S')^2 - p^2 = 0$$

This equation is known as Hamilton-Jacobi equation.

(c) Show that the Hamilton-Jacobi equation yields the solutions $e^{\pm i\int p(x)dx}$.

(d) Obtain the additional factor $\frac{1}{\sqrt{p(x)}}$ in the WKB solutions by going to the next-order approximation.

Solution:

(a)

$$\begin{aligned} y &= e^{iS} \\ y' &= iS'e^{iS} \\ y'' &= iS''e^{iS} - (S')^2 e^{iS} \end{aligned}$$

Plugging those in, we obtain

$$iS''e^{iS} - (S')^2 e^{iS} + p^2 e^{iS} = [iS'' - (S')^2 + p^2]e^{iS} = 0$$

Hence, if $y \neq 0$,

$$iS'' - (S')^2 + p^2 = 0 \tag{4}$$

(b) We don't know what S is, in the first place. So how can one show that the term iS'' can be dropped, that is, it is negligible.The strategy is that we neglect the term iS'', and solve for S. After then, we turn back, and check if we did something sensible with neglecting iS''.

Solving

$$(S')^2 - \lambda^2 P^2 = 0$$

we obtain

$$S = \pm\lambda \int P(x)dx = O(\lambda) \tag{5}$$

Hence

$$\begin{aligned} S' &= \lambda P(x) \\ S'' &= \lambda P'(x) \end{aligned}$$

Thus

$$\underbrace{\frac{1}{\lambda}(i\frac{1}{\lambda}S)''}_{O(\frac{1}{\lambda})} - \underbrace{(\frac{1}{\lambda}S')^2}_{O(1)} + \underbrace{P^2}_{O(1)} = 0$$

Hence, the solution S given by (5) almost satisfies the given equation. So the term iS'' is negligible-compared with $(S')^2$.

(c) Putting (5) into $y = e^{iS}$, we obtian $y = e^{\pm i \int p(x)dx}$.

(d) Let

$$S = \pm\lambda \int P(x)dx + Q(x)$$

where $Q(x)$ is the next-order correction to S.Then

$$\begin{aligned} S' &= \pm\lambda P(x) + Q'(x) \\ S'' &= \pm\lambda P'(x) + Q''(x) \end{aligned}$$

Plugging those into (4), we obtain

$$-\lambda P' - \lambda^2 P^2 - 2i\lambda PQ' + (Q')^2 + \lambda^2 P^2 = 0$$

or

$$P' + 2iPQ' = \frac{1}{\lambda}(Q')^2$$

Neglecting the right-hand side of the last equation, we obtain

$$Q' = i\frac{1}{2}\frac{P'}{P}$$

This integrates to give us

$$Q = i \ln P$$

Thus

$$\begin{aligned} y &= e^{iS} = e^{i[\pm\lambda \int P(x)dx + Q(x)]} \\ &= e^{\pm i \int p(x)dx - \frac{1}{2}\ln p} \\ &= \frac{1}{\sqrt{p(x)}} e^{\pm i \int p(x)dx} \end{aligned}$$

Q.E.D.