

SPERNER'S LEMMA AND FAIR DIVISION PROBLEMS

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ABSTRACT. This paper is a seminar paper that summarizes Francis Edward Su's paper "Rental harmony: Sperner's lemma in fair division" [1] in 1999. In this paper we introduce and prove Sperner's Lemma. Then, we introduce the fair division problem and an algorithm based on Sperner's lemma that can solve some variants of the fair division problem.

1. INTRODUCTION

Many real-life situations require dividing resources and costs among people, such as dividing profit between companies, or paying rent of a shared apartment. In mathematics, such problems have also been studied since long ago. In particular, mathematicians have developed several methods to find an *envy-free division*, which is a division that every person does not envy any other person, i.e. everyone think he/she gets the best (or one of the equally best) option among all people.

We will introduce one of those methods developed by Forrest Simmons in 1980 that gives an envy-free solution to the cake-cutting problem, a variant of fair division problem. This method is based on Sperner's lemma, a famous combinatorial lemma proved by Emanuel Sperner in 1928.

2. SPERNER'S LEMMA

First, we will start by stating and proving Sperner's lemma. This lemma holds for every dimension, but the most common and simplest one is the two-dimensional case.

2.1. Two-dimensional (triangle) Case. In a two-dimensional space (plane), a triangulation of a polygon is a subdivision of the polygon into disjoint triangles. We also call these triangles *elementary triangles*.

Consider any triangulation of a triangle T . We want to label every vertex with one of three colors such that

- (1) The three vertices of T must all have different colors.

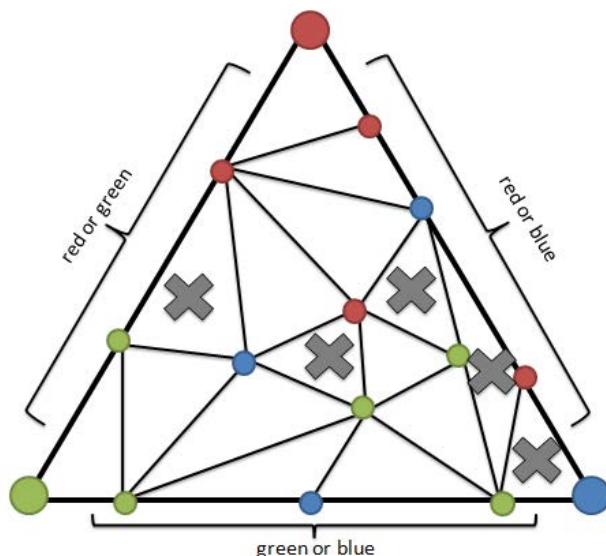


FIGURE 1. An example of Sperner labeling in a triangle. A cross denotes each room with all vertices having different colors.

- (2) Every vertex on the edge of T must have the same color as one of the endpoints of that edge.
- (3) The vertices in the interior of T can have any color.

We call a label that satisfies those three conditions a *Sperner labeling*. The Sperners' lemma states that:

Lemma 2.1. *Any Sperner-labeled triangulation of a triangle must contain an odd number of elementary triangles having all different labeled colors.*

Note that, in many applications of the lemma, we only use the result that there is at least one such triangle, as stated in the following corollary.

Corollary 2.2. *Any Sperner-labeled triangulation of a triangle must contain at least one elementary triangle having all different labeled colors.*

2.2. General Case. Sperner's lemma also applies to a general case in n -dimensional space. First, we define an n -simplex, which is an n -dimensional shape in a similar sense to a triangle.

Definition 2.3. An n -simplex is an n -dimensional shape that consists of $n + 1$ vertices, and an edge joining every pair of vertices.

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 Please see Figure 3. Francis Edward Su. Rental harmony:
 Sperner's lemma in fair division. *American Mathematical
 Monthly*, 106(1999), 930-942.

FIGURE 2. An example of Sperner labeling in a tetrahedron with labeled color 1, 2, 3, and 4. (This figure is taken from Figure 3 in Su's paper [1].)

For example, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron (triangular pyramid).

Definition 2.4. In n -dimensional space, a triangulation of an n -dimensional polytope is a subdivision of the polytope into disjoint n -simplices. We call these n -simplices *elementary n -simplices*.

We define a Sperner labeling on an n -simplex P to be a label such that

- (1) All of the n vertices of P must have different colors.
- (2) Every vertex on the k -dimensional face of P must have the same color as one the corners of that face.
- (3) The vertices in the interior of P can have any color.

For example, Figure 2 shows an example of the three-dimensional case of a Sperner-labeled tetrahedron. Vertices on the edge with endpoints labeled 1 and 2 can be labeled by either color 1 or 2, while vertices on the triangular face with corners labeled 1, 2, and 3 can be labeled by either color 1, 2, or 3.

The general case of Sperners' lemma states that:

Lemma 2.5. *Any Sperner-labeled triangulation of an n -simplex must contain an odd number of elementary n -simplices having all different labeled colors.*

Like in the two-dimensional case, we mainly use the result that there is at least one such n -simplex, as stated in the following corollary.



FIGURE 3. A figure showing the proof of Sperner's lemma in one-dimensional case.

Corollary 2.6. *Any Sperner-labeled triangulation of an n -simplex must contain at least one elementary n -simplex having all different labeled colors.*

3. PROOF OF SPERNER'S LEMMA

We will prove by induction on the number of dimensions. To better clarifying ideas of the proof, we will show the base cases of both one and two-dimension cases, since the ideas of the proof in n -dimensional case is very similar to the two-dimensional case.

3.1. One-dimensional (line segment) Case.

Proof. Suppose that the left and right endpoint of the segment are labeled red and blue, respectively. By considering the parity of color changing when moving from the leftmost to the rightmost vertex, we can conclude that the total number of color changes is odd. Thus, the number of edges with endpoints having different colors is also odd. \square

3.2. Two-dimensional (triangle) Case.

Proof of Lemma 2.1. Suppose we have a Sperner-labeled triangulation of T . We will call each elementary triangle *good* if its three vertices all have different colors.

We define a *door* to be an edge having two endpoints green and blue. First, observe that there is an odd number of doors on the boundary of T (because only one side can have doors, and we proved in the one-dimensional case that the number of such edges is odd). Moreover, each elementary triangle can have at most two doors, and it will be good if and only if it has exactly one door.

We consider the planar graph of T , with each elementary triangle as a vertex (and an outside face as a special vertex v_0) and each room as an edge. Since the degree of v_0 is odd, the number of vertices with odd degree besides v_0 must be odd. Thus, the number of good triangles is odd. \square

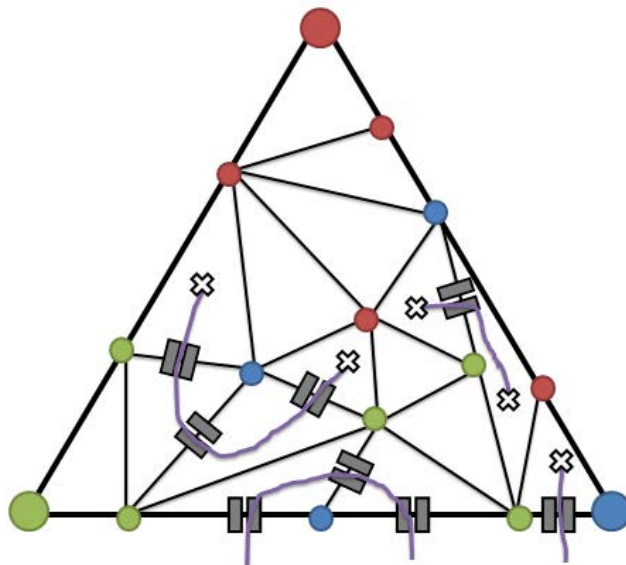


FIGURE 4. A figure showing the proof of Sperner's lemma in two-dimensional case.

3.3. **n -dimensional (n -simplex) Case.** The proof of the general case is similar to the two-dimensional case, as follows:

Proof of Lemma 2.5. Suppose we have a Sperner-labeled triangulation of an n -dimensional polytope P . Similar to the two-dimensional case, we call each elementary n -simplex *good* if its vertices all have different colors.

Suppose we use color 1, 2, ..., n in labeling. We define a *door* to be an $(n - 1)$ -simplex having corners labeled 1, 2, ..., $n - 1$ in some permutation. From the hypothesis of induction in the $(n - 1)$ -dimensional case, there is an odd number of doors on the boundary of P (because only one face can have doors, and we proved in the $(n - 1)$ -dimensional case that the number of such simplices is odd). Moreover, each elementary n -simplex can have at most two doors, and it will be good if and only if it has exactly one door.

We consider a graph of P , with each elementary n -simplex as a vertex (and an outside face as a special vertex v_0) and each room as an edge. Since the degree of v_0 is odd, the number of vertices with odd degree besides v_0 must be odd. Thus, the number of good n -simplex is odd. \square

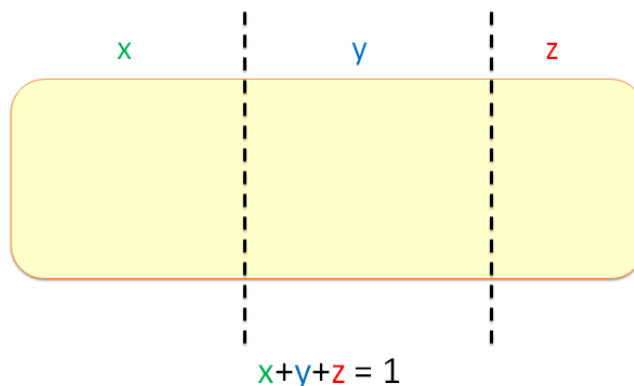


FIGURE 5. An example of a cut-set in the case of three people.

4. CAKE-CUTTING PROBLEM

We now go back to the fair division problem mentioned in the introduction. The cake-cutting problem is a variant of that problem which focuses on finding an envy-free division of a rectangular cake among n people. Note that in the case that everyone has exactly the same perspective on the size of the cake, we can simply equally divide the cake into n equal pieces and give each piece to each person. However, this problem is much more complex because each person may have a different perspective on the size of each piece of cake, e.g. one person may prefer piece A to piece B , but an other person may prefer piece B to piece A .

4.1. Simmons' Algorithm. In 1980, Forrest Simmons developed an algorithm to find an envy-free division of the cake-cutting problem. We consider any $n - 1$ vertical cuts parallel to the side of the cake, dividing the cake into n pieces. Such a way of cutting is called a *cut-set*. First, observe that in a particular cut-set, if we ask all players which part do they prefer and get all different answers, then that cut-set gives an envy-free division (because every one gets the piece he/she prefer the most). We will use Sperner's lemma to prove that there must exists such cut-set, as well as finding one.

Note that, in order for Simmons' algorithm to work, the following two assumptions are required:

- (1) **Hungry Players:** Every player prefers any non-empty piece to an empty piece.
- (2) **Closed Preference Set:** If a player prefers the convergent sequence of cut-set, he/she also prefers the limiting cut-set.

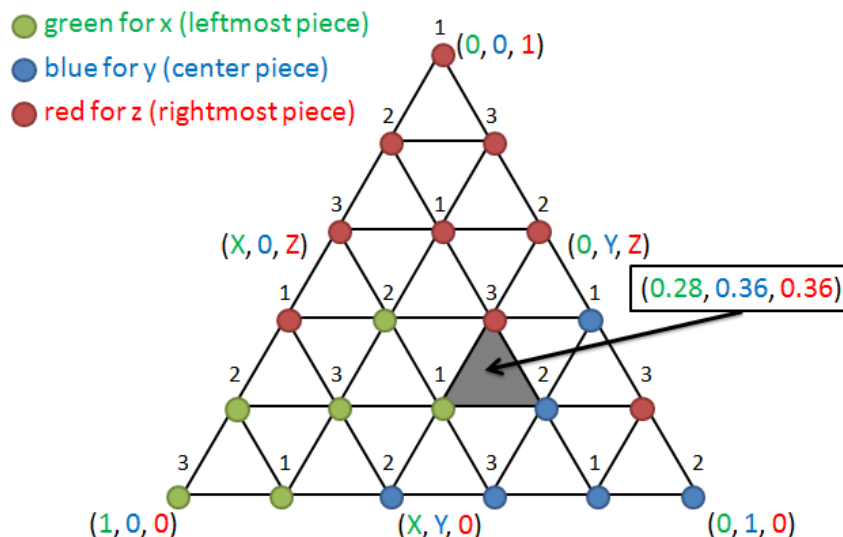


FIGURE 6. An example of constructing a labeling in a triangle with each vertex representing each cut-set for three people.

This assumption can be stated in other words that, for a small enough ϵ , moving the cuts by ϵ does not change the preference of any player.

Simmons' Algorithm works as follow:

Consider every possible cut-set that divides the cake into n (maybe empty) pieces. Let x_i be the proportion of i th leftmost part. We have $x_1 + x_2 + \dots + x_n = 1$ and $x_i \geq 0$ for every $i = 1, 2, \dots, n$.

In the n -dimensional space, consider a polytope formed by linear programming $x_1 + x_2 + \dots + x_n = 1$ and $x_i \geq 0$ and $x_i \geq 0$ for every $i = 1, 2, \dots, n$. The resulted polytope is a regular $(n - 1)$ -simplex.

Then, we triangulate that simplex into smaller regular $(n - 1)$ -simplices with each having side length less than ϵ , for a small enough ϵ , as well as writing numbers $1, 2, \dots, n$ on the vertices in a way that every elementary $(n - 1)$ -simplex has vertices with all different numbers.

An example of construction in the case of three people is shown in Figure 6. In that figure, the resulted polytope is a regular triangle (regular 2-simplex). We then divide it into k^2 smaller regular triangles, for some integer k as big as we want. Then, we write a number on each vertex $1, 2, 3$ in a cyclic order such that every elementary triangle has all three vertices with different numbers.

Then, at each vertex with coordinates (x_1, x_2, \dots, x_n) with number i , we ask player i that which piece of cake that he/she prefers if the sizes of pieces of cake is x_1, x_2, \dots, x_n , respectively. For example, at the top vertex in Figure 6, we ask player 1 that if the sizes of three pieces are 0, 0, and 1, respectively, which piece does he/she prefer. We then color that vertex according to the answer.

Observe that, in each of the k -dimensional face of the polytope, one or more of the coordinates must be zero. For example, the y -coordinate along the left edge of the triangle in Figure 6 is always zero. By the hunger players assumption, no people prefer the piece with size zero, so the color of vertices on each face must be the same as one of the corners of that face. Therefore, the color labeling of vertices in the simplex is a Sperner labeling.

By Sperners' lemma, there must be at least one elementary $(n - 1)$ -simplex that has vertices with different colors. We then divide the cake by the cut-set represented by any interior point of that $(n - 1)$ -simplex. Since the size of each elementary simplex is less than ϵ , by the closed preference set assumption, all people will be satisfied with that cut-set.

5. RENT-PARTITIONING PROBLEM

A rent-partitioning problem is another variant of the fair division problem. This problem originated from the situation when n people rent a shared apartment with n different bedrooms. The problem is to determine which bedroom should each person get, and how much rent each person should pay.

One possible attempt is to consider every possible assignment of price to each bedroom. i.e. set the price of room i to be x_i such that $x_1 + x_2 + \dots + x_n = S$, when S is the total price of the apartment, and $x_i \geq 0$ for every $i = 1, 2, \dots, n$. Then, we consider the $(n - 1)$ -simplex obtained from the linear programming and color it like in the cake-cutting problem. However, this problem is different from the cake-cutting problem in one aspect. In the cake-cutting problem, the bigger the piece of cake is, the higher chance people will want it; however, in this problem, the higher the price of a room is, the lower chance people will want that room. In fact, the label in the $(n - 1)$ -simplex in this case should be the opposite from a Sperner labeling. For example, Figure 7 shows an example of labeling obtained from the rent-partitioning problem with three people. Vertices on the edge can be labeled with only one color, while the three vertices at the corner of the big triangle can be labeled with one of two colors.

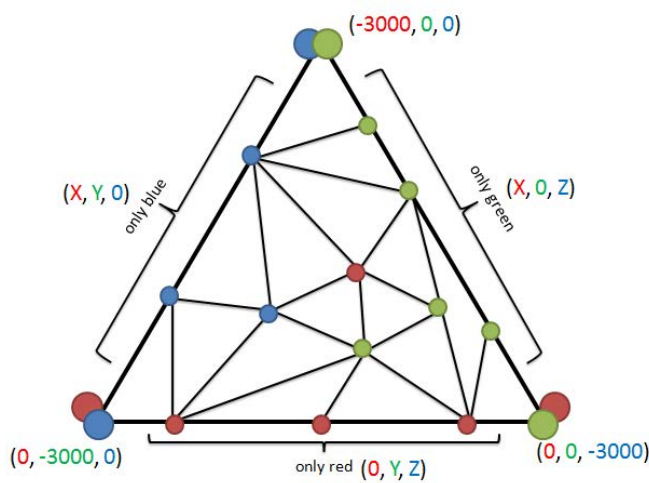


FIGURE 7. An example of a labeling obtained from the rent-partitioning problem with three people.

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FIGURE 8. An example of the dualization of a triangle. (This figure is taken from Figure 10 in Su's paper [1].)

One method to solve this issue is to dualize the simplex. Each k -dimensional face will become an $(n - 1 - k)$ -dimensional face, as shown is figure 8. In fact, Herbert E. Scarf have proven that the dual simplex satisfies Sperner labeling properties.

There is another simple way of solving the rent-partitioning problem without using the complicated dualization. This can be done by each person first paying the full rent price and then assigning to each room a “rebate” price that a person who takes that room will get. Then, we can use Simmons' Algorithm to assign rooms to people in the same way as in the cake-cutting problem. However, in some extreme cases that there is a room that nobody wants it even if it is free, someone

may end up with that room with a negative price. Such cases should not happen too often, though, in real-world situations.

6. CONCLUSION

We have proved Sperner's Lemma for the n -dimensional case. Then, using Sperner's lemma, we have proved Simmon's algorithm that can find envy-free division for n people to the cake-cutting problem, a variant of the fair division problem, as well as modifying to solve the rent-partitioning problem, another variant of the fair division problem.

REFERENCES

- (1) Francis Edward Su. Rental harmony: Sperner's lemma in fair division. *American Mathematical Monthly*, 106(1999), 930-942.

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