Lecture 2

Definition of Green's function for general domains.

Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then for $y \in \Omega$, the Green Representation formula tells us

$$u(y) = \int_{\partial\Omega} (u\frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y)\frac{\partial u}{\partial \nu})d\sigma + \int_{\Omega} \Gamma(x-y)\Delta u dx.$$

Definition 1 *For integrable f, $\int_\Omega \Gamma(x-y)f(x)dx$ is called Newtonian Potential with density f.*

Remark 1 *If $u \in C_0^2(\mathbb{R}^n)$, i.e. compact supported, then have*

$$u(y) = \int_{\Omega} \Gamma(x-y)\Delta u dx.$$

If u is harmonic, then we have

$$u(y) = \int_{\partial\Omega} (u\frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y)\frac{\partial u}{\partial \nu})d\sigma.$$

Thus harmonic functions are analytic.

Now let h be harmonic, by Green's 2^{nd} identity, we get

$$\int_\Omega h\Delta u = \int_{\partial\Omega} (h\frac{\partial u}{\partial \nu} - u\frac{\partial h}{\partial \nu})ds$$

i.e.

$$0 = \int_{\partial\Omega} (u\frac{\partial h}{\partial \nu} - h\frac{\partial u}{\partial \nu})ds + \int_\Omega h\Delta u$$

Adding Green's representation formula, we get

$$u(y) = \int_{\partial\Omega} \{(u(x)(\frac{\partial}{\partial \nu_x}\Gamma(x-y) + \frac{\partial h}{\partial \nu_x}) - (\Gamma(x-y)+h(x))\frac{\partial u}{\partial \nu_x})ds\} + \int_\Omega (\Gamma(x-y)+h(x))\Delta u dx.$$

Now fix x, we choose $h_y(x)$ s.t. $\Delta h_y(x) = 0$ in Ω and $h_y(x) = -\Gamma(x-y)$ on $\partial\Omega$. Let $G(x,y) = \Gamma(x-y) + h_y(x)$, then we have

$$u(y) = \int_{\partial\Omega} u(x)\frac{\partial}{\partial \nu_x}G(x,y)ds + \int_\Omega G(x,y)\Delta u dx.$$

Definition 2 *Such a function G(x,y), defined for $x \in \Omega, y \in \overline{\Omega}, x \neq y$ which satisfies $G(x,y) = 0$ for $x \in \partial\Omega$ and $h(x,y) = G(x,y) - \Gamma(x-y)$ is harmonic in $x \in \Omega$, is called a Green function for domain Ω.*

Remark 2 *1. By Maximum Principle, G is unique if exists.*
2. If G exists for a domain Ω and u is harmonic in Ω, then we can get an explicit formula for u in terms of boundary values:

$$u(y) = \int_{\partial\Omega} u \frac{\partial G}{\partial \nu} ds.$$

Green's function for ball $B(0,R)$

Proposition 1 *The Green's function for the ball $B(0,R)$ is*

$$G(x,y) = \begin{cases} \frac{1}{n(2-n)\omega_n}(|x-y|^{2-n} - |\frac{R}{|x|}x - \frac{|x|}{R}y|^{2-n}) & , \quad n \geq 3, \\ \frac{1}{2\pi}(\log|x-y| - \log|\frac{R}{|x|}x - \frac{|x|}{R}y|) & , \quad n = 2. \end{cases}$$

Remark 3 $G(x,y) = \Gamma(x-y) - \Gamma(\frac{R}{|x|}x - \frac{|x|}{R}y)$*, thus* $\Delta_y G(x,y) = 0$ *and* $G(x,y) = \Gamma(x-y) +$ *a harmonic function on boundary.*

Claim 1 $G(x,y) = G(y,x), G(x,y) \leq 0.$

Proof: By squaring, we can get $|\frac{R}{|y|}y - \frac{|y|}{R}x| = |\frac{R}{|x|}x - \frac{|x|}{R}y|$, thus $G(x,y) = G(y,x)$. This implies $\Delta_x G(x,y) = 0$ by previous remark.
For $x, y \in B(0,R)$, we have $|x-y| \leq |\frac{R}{|x|}x - \frac{|x|}{R}y|$, thus $G(x,y) \leq 0$ since the function Γ is decreasing as a real function. ■

Proposition 2 $\frac{\partial G}{\partial \nu_x} = \frac{R^2 - |y|^2}{n\omega_n R} \frac{1}{|x-y|^n}, x \in \partial B(0,R)$

Proof: By symmetry, $G(x,y) = \frac{1}{n(2-n)\omega_n}(|x-y|^{2-n} - |\frac{R}{|y|}y - \frac{|y|}{R}x|^{2-n})$. Thus

$$\frac{\partial G}{\partial x_i} = \frac{1}{n\omega_n}(\frac{x_i - y_i}{|x-y|^n}) - \frac{(\frac{Ry_i}{|y|} - \frac{|y|}{R}x_i)(\frac{-|y|}{R})}{|x-y|^n}.$$

So

$$\begin{aligned} \frac{\partial G}{\partial \nu_x} =< \frac{\partial G}{\partial x_i}, \frac{x_i}{|x|} > &= \frac{1}{n\omega_n}\frac{1}{|x-y|^n}(\frac{1}{|x|})(|x|^2 - < x,y > + < x,y > - \frac{|y|^2}{R^2}|x|^2) \\ &= \frac{1}{n\omega_n R|x-y|^n}(R^2 - |y|^2) \end{aligned}$$

This completes the proof. ■

Corollary 1 *If* $u \in C^2(B_R) \cap C^0(\overline{B_R})$ *and* $\Delta u = 0$*, then*

$$u(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R} \frac{u(x)}{|x-y|^n} d\sigma_x.$$

Remark 4 *Previously, we regarded $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Under the assumption of this corollary, we can get formula holds for $r < R$. Since $u \in C^0(\overline{\Omega})$, just take limit as $r \to R$.*
Again, we see that harmonic functions are analytic.

Poisson Integral Formula

Theorem 1 *Let $\varphi : \partial B(0,R) \to \mathbb{R}$ be continuous, then*

$$u(x) = \begin{cases} \frac{R^2-|x|^2}{n\omega_n R} \int_{\partial B(0,R)} \frac{\varphi(y)}{|x-y|^n} d\sigma_y & , \quad x \in B(0,R), \\ \varphi(x) & , \quad x \in \partial B(0,R). \end{cases}$$

satisfies $\Delta u = 0$ in $B(0,R)$ and $u \in C^2(B) \cap C^0(\bar{(}B))$

Proof: For $x \in B(0,R)$, the definition of u gives $u(x) = \int_{\partial B(0,R)} \varphi(y) \frac{\partial G}{\partial \nu_y}(x,y) d\sigma_y$, thus

$$\begin{aligned} \Delta_x u(x) &= \int_{\partial B(0,R)} \varphi(y) \Delta_x \frac{\partial G}{\partial \nu_y}(x,y) d\sigma_y \\ &= \int_{\partial B(0,R)} \varphi(y) \frac{\partial}{\partial \nu_y} \Delta_x G(x,y) d\sigma_y = 0. \end{aligned}$$

so $\Delta u(x) = 0$ in B and $u \in C^2(B)$
We have known that for harmonic function $\omega \in C^2(B) \cap C^1(\overline{B_R})$,

$$\omega(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B(0,R)} \frac{\omega(x)}{|x-y|^n} d\sigma_x.$$

Take $\omega \equiv 1$, we get $1 = \frac{R^2-|y|^2}{n\omega_n R} \int_{\partial B(0,R)} \frac{1}{|x-y|^n} d\sigma_x$, i.e.

$$1 = \int_{\partial B(0,R)} \frac{R^2 - |y|^2}{n\omega_n R} \frac{1}{|x-y|^n} d\sigma_x = \int_{\partial B} K(x,y) d\sigma_y.$$

Here $K(x,y) = fracR^2 - |y|^2 n\omega_n R \frac{1}{|x-y|^n}$ is called Poisson Kernel.
Now consider $x_0 \in \partial B$. For any $\epsilon > 0$, there $\exists \delta > 0$ s.t. $|\varphi(x) - \varphi(x_0)| < \epsilon$ for any $|x - x_0| < \delta$. Choose M large enough such that $\varphi(x) < M \forall x \in \partial B$. For $|x - x_0| < \frac{\delta}{2}$, we have

$$\begin{aligned} |u(x) - u(x_0)| &= |\int_{\partial B} K(x,y)(\varphi(y) - \varphi(x_0)) d\sigma_y| \\ &\le \int_{|y-x_0|\le\delta} K(x,y)|(\varphi(y) - \varphi(x_0))| d\sigma_y + \int_{|y-x_0|>\delta} K(x,y)|(\varphi(y) - \varphi(x_0))| d\sigma_y \\ &\le \epsilon + 2M \frac{R^2 - |x|^2}{n\omega_n R} \frac{1}{(\delta/2)^n} n\omega_n R^{n-1} \\ &\le \epsilon + 2C(R^2 - |x|^2). \end{aligned}$$

Thus for x close to ∂B, $|u(x) - u(x_0)| \le 2\epsilon$, i.e. $x \in C^0(\overline{B})$ ■

Mean Value Property (MVP)

Theorem 2 *If a $C^0(\Omega)$ function u satisfies*

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u d\sigma$$

*for every ball $B = B(y, R) \subset\subset \Omega$ (**MVP**), then u is harmonic. In particular, u is analytic.*

Proof: Take any $B(y, R) \subset\subset \Omega, u \in C^0(\partial B(y, R))$. Thus by Poisson integral formula, there is harmonic function h on $B(y, R)$ s.t. $h = u$ on $\partial B(y, R)$.

Consider $\omega = h - u$. Obviously ω satisfies MVP on any ball $\subset B(y, R)$. Recall that our maximum principle and uniqueness proof only need MVP, so ω has zero boundary value implies $\omega = 0$ in $B(y, R)$. So $u = h$ in B, i.e. u is harmonic. ■

Remark 5 *The proof just need "for each $x \in \Omega$, $\exists B(x, R) \subset \Omega$ s.t. MVP is satisfied on all balls in $B(x, R)$".*

counterexample *(NOT C^0):Take u on plane, $u(x, y) = 1$ for $y > 0$, $u(x, y) = -1$ for $y < 0$, $u(x, y) = 0$ for $y = 0$. Obviously u is not harmonic.*

Corollary 2 *The limit of a uniformly convergent sequence of harmonic functions is harmonic.*

Proof: The limit is continuous and still satisfies MVP. ■