Lecture Notes for LG's Diff. Analysis

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1 Schauder Estimate

Recall the following proposition:

Proposition 1.1 (Baby Schauder). If $0 < \lambda \le a_{ij} \le \Lambda$, $0 < \alpha < 1$, $[a_{ij}]_{C^\alpha} \le B$, $Lu = 0$, then

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \lesssim \|u\|_{C^2(B_1)}.$$

We would like to get that C^2 norm down to a C^0 norm. To do this, we use the following proposition:

Proposition 1.2. If $u \in C^2(\Omega)$, then for all $\epsilon > 0$,

$$\|u\|_{C^2(\Omega)} \lesssim \epsilon[\partial^2 u]_{C^\alpha(\Omega)} + C_1 \epsilon^{-C_2} \|u\|_{C^0(\Omega)}$$

where $C_1, C_2 > 0$ depend on Ω and α.

Proof. We will just prove $\|\partial^2 u\| \le RHS$.

By scaling, we can assume that $\|\partial^2 u\|_{C^0} = 1$ and by rotating, we get that at some point $x_0 \in \Omega \cup \partial\Omega$, $|\partial_1^2 u(x_0)| = 1$.

If $[\partial^2 u]_{C^\alpha(\Omega)} > \epsilon^{-1}$ we are done. So assume otherwise. Pick r such that $r^\alpha \epsilon^{-1} < \frac{1}{100}$. Then for all $x \in B(x_0, r)$, $|\partial_1^2 u(x)| \ge 99/100$.

Now, there exists a segment $\sigma \subset \Omega$ of length r in the x_1 direction (if not, just pick r to be smaller). On this segment, $\partial_1 u$ is monotone. Therefore, there exists a smaller segment $\sigma_1 \subset \sigma$ of length $\geq r/4$ such that on this segment, $|\partial_1 u| \geq r/5$. By the same argument, there exists a still smaller segment $\sigma_2 \subset \sigma_1$ on which $|u(x)| \geq r^2/1000$. Therefore,

$$\|u\|_\infty \geq \frac{r^2}{1000} \geq C\epsilon^{2/\alpha}$$

where C depends only on r (which in turn depends only on Ω). So, picking $C_1 = 1/C$ and $C_2 = 2/\alpha$, we get our desired estimate. $\square$

So, we now have that

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\epsilon \|u\|_{C^{2,\alpha}(B_1)} + \frac{C}{\epsilon}\|u\|_{C^0(B_1)}$$

We would like to use a rearrangement trick to get the $\|u\|_{C^{2,\alpha}(B_1)}$ from the right hand side to the left hand side, but the two norms are over different balls, so we have to be more clever than that. Towards this, we note the following simple extension of the baby Schauder estimate.

Proposition 1.3 (Baby Schauder Prime)**.** Given the assumptions of Schauder, and $w \in [0, 1/8]$,

$$\|u\|_{C^{2,\alpha}(B_{\frac{3}{4}-2w})} \leq Cw^{-A}\|u\|_{C^2(B_{\frac{3}{4}-w})}$$

for $A > 0$.

Proof. If we go back through our proof of Baby Schauder, and trace where the w comes in, we find at the end all that pops out is this power of w. As expected, as w gets smaller, our estimate gets worse. $\square$

Now let's use this to make our fake rearrangement into an actual rearrangement. Define the following function:

$$F(w) = \rho(w)\|u\_C^{2,\alpha}(B_{3/4-2w})$$

where ρ is continuous, $\rho(0) = 0$, and $\rho(w) > 0$ for $w > 0$ - the exact equation for ρ will be determined later. Then certainly $F \in C^0([0, 1/8])$, $F(0) = 0$, and since F is continuous over a compact interval, F attains its maximum at some $w_0 \neq 0$. Our goal will be to prove an upper bound for $F(w_0)$ in terms of $\|u\|_{C^0(B_1)}$. Let's begin.

$$\begin{aligned}
F(w_0) = \rho(w_0)\|u\|_{C^{2,\alpha}(B_{3/4-2w_0})} &\leq C\rho(w_0)\|u\|_{C^2(B_{3/4-w_0})}w_0^{-A} \\
&\leq C\rho(w_0)w_0^{-A}\left(\epsilon\|u\|_{C^{2,\alpha}(B_{3/4-w_0})} + \epsilon^{-C_2}\|u\|_{C^0(B_1)}\right) \\
&\leq C\epsilon w_0^{-A}\frac{\rho(w_0)}{\rho(w_0/2)}F(w_0/2) + Cw_0^{-A}\rho(w_0)\epsilon^{-C_2}\|u\|_{C^0(B_1)} \\
&\leq C\epsilon w_0^{-A}\frac{\rho(w_0)}{\rho(w_0/2)}F(w_0) + Cw_0^{-A}\rho(w_0)\epsilon^{-C_2}\|u\|_{C^0(B_1)}
\end{aligned}$$

Now, we pick ϵ so that we can legit use rearrangement, that is,

$$\epsilon = \frac{w_0^A\rho(w_0/2)}{2C\rho(w_0)}$$

and so

$$F(w_0) \leq (2C)^{1+C_2}w_0^{-A-AC_2}\rho(w_0)\left(\frac{\rho(w_0)}{\rho(w_0/2)}\right)^{C_2}\|u\|_{C^0(B_1)}.$$

Thus, if we let $\rho(w) = w^B$ for $B > 2C_2A$, we get that

$$F(w_0) \leq Cw^K2^{C_2B}\|u\|_{C^0(B_1)} \leq C\|u\|_{C^0(B_1)}.$$

Finally, we have

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq \frac{1}{\rho(1/8)} F(w_0) \lesssim \|u\|_{C^0(B_1)},$$

completing our proof of Schauder Regularity.

2 "Solving" Elliptic PDE

We begin with the Dirichlet problem:

Theorem 2.1. If L obeys the conditions of the Schauder estimate (**Proposition 1.1**) on B_1, $\phi \in C^{2,\alpha}(\partial B_1)$, then there is a unique $u \in C^{2,\alpha}(B_1)$ such that $Lu = 0$, and $u = \phi$ on ∂B_1.

Note: uniqueness follows from the maximum principle.

History lesson: In the 1800's, Poisson found an explicit formula if $L = \Delta$:

$$u(x) = (1 - |x|^2) \int_{\partial B_1} \frac{\phi(y)}{|x - y|^n} dArea_y$$

As setup, consider to Banach spaces X and Y, and $L : X \to Y$ a bounded linear map.

Definition 2.1. L is an isomorphism if there exists L^{-1} which is bounded and linear. Equivalently, L is surjective and there exists $\lambda > 0$ such that $\lambda \|x\|_X \leq \|Lx\|_Y$.

With this notation, we have

Proposition 2.1. Let $L_1 : X \to Y$ be an isomorphism with lower bound $\lambda > 0$. Then if $\|L_2 - L_1\|_{oper} \leq \mu < \lambda$, L_2 is also an isomorphism.

Proof. Clearly L_2 satisfies the lower bound property, as

$$\|L_2(x)\| \geq \|L_2(x)\| - \|(L_2 - L_1)x\| \geq (\lambda - \mu)\|x\|$$

It remains to show that L_2 is surjective. Let $y \in Y$. We will use an iterative procedure to find x such that $L_2 x = y$.

Take $x_1 \in X$ such that $L_1 x_1 = y_1 = y$. Then

$$\|y - L_2 x_1\| = \|y - L_1 x_1 + (L_1 - L_2)x_1\| \le \mu\|x_1\| \le \frac{\mu}{\lambda}\|y\| < \|y\|$$

Now iteratively define $y_{j+1} = y_j - L_2 x_j$ and $x_j = L_1^{-1} y_j$. We have that $\|y_{j+1}\| \le \frac{\mu}{\lambda}\|y_j\|$ and $\|x_j\| \le \lambda^{-1}\|y_j\|$, which goes to zero exponentially fast in j. Thus, we can define $x := \sum_{j=1}^\infty x_j$, and

$$L_2 x = \sum_{j=1}^\infty (y_j - y_{j+1}) = y_1 = y$$

$\square$

We immediately get

Corollary 2.1. If $L_t : X \to Y$ is a continuous set of operators, and L_0 is an isomorphism and $\lambda\|x\| \le \|L_t x\|$ for all t, then L_t is an isomorphism for all t.

Proof. take a sequence $0 = t_0 < t_1 < \cdots < t_{final}$ such that $\|L_{t_{j+1}} - L_{t_j}\| < \lambda$, and then apply the previous proposition. $\square$

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