18.156 Lecture Notes

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Today, we'll finish up the proof of the Calderon-Zygmund theorem and see some examples.

Part IV: Duality. Recall that we have already proven CZ for $1 < p \leq 2$. Now, let p be such that $2 < p < \infty$ and let p' be the dual exponent (so $1 < p < 2$). Then, we have that

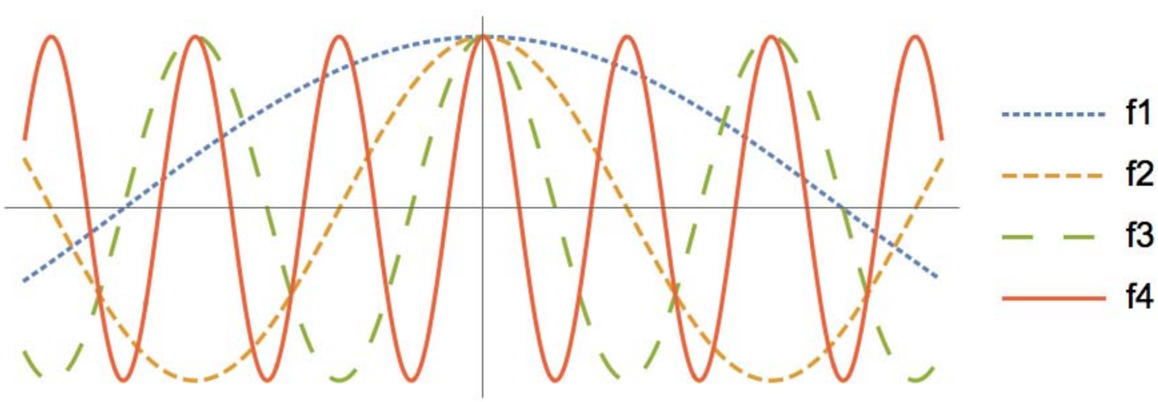
$$\begin{aligned}
\|Tf\|_p &= \sup_{\|g\|_{L^{p'}} \leq 1} \int Tf \cdot g \\
&= \sup \int \int f(y) K(x-y)\, dy\, g(x)\, dx \\
&= \sup \int f(y) \int K(x-y) g(x)\, dx\, dy \\
&= \sup \int f \cdot (\overline{K} * g) \\
&\leq \|f\|_p \cdot \sup \|\overline{T} g\|_{L^{p'}} \\
&\lesssim \|f\|_p,
\end{aligned}$$

by CZ for p'. We note here that $\overline{K}(x)$ is defined as $K(-x)$ and $\overline{T}g = g * \overline{K}$. This completes the proof the the Calderon-Zygmund theorem.

Let us look at an application of CZ now. Suppose that $f_k : \mathbb{R}^d \to \mathbb{C}$,

$$\text{supp } \hat{f}_k \subset A_k = \{\omega : 2^{k-1} \leq |\omega| \leq 2^{k+1}\},$$

and $f = \sum f_k$. We'll call this condition (*).



The intuition here is that the f_k should be almost independent. That is, knowing that $f_k(x) \in [a, b]$ shouldn't tell you that much about the value of $f_\ell(x)$ for $\ell \neq k$. If the f_k were indeed independent, then

$$\left|\sum f_k\right| \sim \left(\sum |f_k|^2\right)^{1/2}$$

with high probability in x. The next theorem says that our intuition is pretty much what happens.

Theorem 1 (Littlewood-Paley)**.** *If (*) holds, then*

$$\|f\|_{L^p} \sim \left\|\left(\sum |f_k|^2\right)^{1/2}\right\|_p$$

(up to a factor $C(p, d)$).

What we'll prove today is the $\lesssim$ in the above theorem. Define

$$T_k g = (\psi_k \hat{g})^\vee.$$

Here, ψ_k is a bump function where $\psi_k = 1$ on A_k, and is supported on $\tilde{A}_k := \{\omega : 2^{k-2} \leq |\omega| \leq 2^{k+2}\}$. We also want ψ_k as smooth as possible, and we can show that we can construct ψ_k so that $|\psi_k| \leq 1, |\partial\psi_k| \leq 2^{-k}, |\partial^2\psi_k| \leq 2^{-2k}$, and so on. We note that $T_k f_k = f_k$.

Now, we define $\vec{g} = (\ldots, g_{-1}, g_0, g_1, \ldots)$ and $\vec{T}\vec{g} = \sum_k T_k g_k$. Note here that $\vec{T}\vec{f} = f$. Then,

$$|\vec{g}(x)| = \left(\sum_k |g_k(x)|^2\right)^{1/2}$$

and

$$\|\vec{g}\|_p = \left\|\left(\sum_k |g_k|^2\right)^{1/2}\right\|_p = \text{RHS of L-P.}$$

Theorem 2. *$\|\vec{T}\vec{g}\|_p \lesssim \|\vec{g}\|_p$ for all $1 < p < \infty$ implies the previous theorem (by taking $\vec{g} = \vec{f}$).*

Notice that $T_k g_k = g_k * \psi_k^\vee$ and

$$\vec{T}\vec{g} = \vec{g} * \vec{\psi}^\vee = \int \vec{g}(y) \cdot \vec{\psi}^\vee(x - y)\, dy = \int \sum_k g_k(x) \psi_k^\vee(x - y)\, dy.$$

In Calderon-Zygmund, our f and K were scalar valued. Now, we have that $\vec{g}, \vec{K}$ are vector valued in ℓ^2. But this turns out not to be an issue since the proof of Calderon-Zygmund applies almost verbatim for vector valued functions.

So to be able to apply Calderon-Zygmund, we need to theck three things:

(i) $(\sum_k |\psi_k^\vee(x)|^2)^{1/2} \lesssim |x|^{-d}$

(ii) $(\sum_k |\partial\psi_k^\vee(x)|^2)^{1/2} \lesssim |x|^{-d-1}$

(iii) $\|\vec{T}\vec{g}\|_2 \lesssim \|\vec{g}\|_2$

Proof. Let's prove each of the three statements above.

(i) We have that $|\psi_k^\vee(x)| \leq \|\psi_k\|_{L^1} \sim 2^{kd}$ and $|\psi_k^\vee(x)| \sim 2^{kd}$ for $|x| \lesssim 2^{-k}$. It also decals rapidly when $|x| \gg 2^{-k}$ by smoothness (which we can prove by integration by parts). So we have that
$$\sum_k |\psi_k^\vee(x)|^2 \lesssim \sum_{|x|\lesssim 2^{-k}} 2^{2dk} + \text{rapidly decreasing terms} \lesssim |x|^{-2d}.$$
Now, we get the bound we want by taking square roots.

(ii) We have that $|\partial\psi_k^\vee(x)| = |(2\pi i\omega\psi_k)^\vee(x)|$. Now, $|\omega\psi_k| \lesssim 2^k$ and $|\partial(\omega\psi_k)| \lesssim 1, \dots$ so $|\partial\psi_k^\vee(x)| \lesssim 2^{k(d+1)}$ on $|x| \lesssim 2^{-k}$ and is rapidly decaying for $|x| \gg 2^{-k}$. Therefore, we have that
$$\sum_k |\partial\psi_k^\vee(x)|^2 \lesssim \sum_{|x|\lesssim 2^{-k}} 2^{2k(d+1)} + \text{rapidly decreasing terms} \lesssim 2^{-2(d+1)}.$$
Again, we can take square roots to get that bound that we want.

(iii) We have that
$$\|\vec{T}\vec{g}\|_2^2 = \|\sum g_k * \psi_k^\vee\|_2^2 = \|\sum \psi_k \cdot \hat{g}_k\|_2^2 = \int |\sum \psi_k \cdot \hat{g}_k|^2.$$
Note now that for all ω, there are ≤ 4 nonzero terms in the above sum. So we have that
$$\|\vec{T}\vec{g}\|_2^2 \lesssim \int \sum_k |\psi_k \hat{g}_k|^2 \leq \sum_k \int |\hat{g}_k|^2 = \int \sum_k |g_k|^2 = \|\vec{g}\|_2^2.$$
Taking square roots give us the bound that we want.

□

From this lemma and the Calderon-Zygmund theorem, we have the Littlewood-Paley theorem.