Lecture notes for class on Friday March 6

1. Non-linear elliptic PDE

We now begin to study non-linear elliptic PDE. We begin here with one of the simplest examples, the equation $\triangle u - u^3 = 0$. We will solve the Dirichlet problem for this equation, and along the way we will prove regularity estimates. To formulate the Dirichlet problem, we consider the following (non-linear!) map.

$$N : C^{2,\alpha}(B_1) \to C^{\alpha}(B_1) \oplus C^{2,\alpha}(\partial B_1),$$

$$Nu := \left(\triangle u - u^3, u|_{\partial B_1} \right).$$

We also let $\alpha \in (0,1)$ denote a Holder exponent.

Theorem 1. *The map N is surjective. In particular, for any $\phi \in C^{2,\alpha}(\partial B_1)$, there is a function $u \in C^{2,\alpha}(B_1)$ so that $\triangle u - u^3 = 0$, and $u|_{\partial B_1} = \phi$.*

By contrast, the equation $\triangle u + u^3 = 0$ is not well-behaved. It is impossible to solve the Dirichlet problem for this equation. Although, we won't rigorously prove this, we will explain why the equations are different.

Along the way we will also prove regularity estimates for solutions of the equation $\triangle u - u^3 = 0$, and we will see how regularity estimates are crucial to proving the existence of solutions.

2. Functional analysis setup

In this section we review some functional analysis that we will use to prove Theorem 1.

We let X denote the Banach space $C^{2,\alpha}(B_1)$ and we let Y denote the Banach space $C^{\alpha}(B_1) \oplus C^{2,\alpha}(\partial B_1)$, so $N : X \to Y$. We will show that N is a C^1 map from X to Y and compute its derivative. Let us first recall what this means.

For any Banach spaces X, Y, a map $F : X \to Y$ is differentiable at a point $x \in X$ if there is some (bounded) linear map $L : X \to Y$ so that

$$\|F(x+h) - F(x) - L(h)\|_Y = o(\|h\|_X) \text{ as } \|h\|_X \to 0.$$

The map L is unique, and we write $dF_x := L$. The map F is C^1 if it is differentiable at each point, and if the derivative dF_x is continuous in x. (The map dF_x sends X to the space $Hom(X,Y)$ of bounded linear operators from X to Y. The space $Hom(X,Y)$ is equipped with the operator norm, making it a normed space. We use this norm when we speak of dF_x being continuous.)

Proposition 2. *The map $N : X \to Y$ is C^1 and its derivative is given as follows:*

$$dN_u(v) = L_u := (\triangle v - 3u^2 v, v|_{\partial B_1}).$$

Remark. We can find the formula for dN_u by "formally" differentiating $\triangle u - u^3$. I.e. we write $\triangle(u + \epsilon v) - 3(u + \epsilon v)^3$, and we keep the terms of degree 1 in ϵ. This yields the formula above, but it doesn't immediately prove that N is differentiable at u with derivative L_u.

Proof. To show that N is differentiable at $u \in X$ with derivative L_u, it suffices to check that

$$\|N(u + v) - N(u) - L_u(v)\|_Y = o(\|v\|_X).$$

The left-hand side works out to

$$(-3uv^2 - v^3, 0).$$

We have to bound

$$\|(-3uv^2 - v^3, 0)\|_Y = \|3uv^2 + v^3\|_{C^\alpha(B_1)}.$$

We note that $\|fg\|_{C^\alpha} \le \|f\|_{C^\alpha}\|g\|_{C^\alpha}$. (Recall that $\|f\|_{C^\alpha} := [f]_{C^\alpha} + \|f\|_{C^0}$, and use the Liebniz rule $[fg]_{C^\alpha} \le \|f\|_{C^0}[g]_{C^\alpha} + [f]_{C^\alpha}\|g\|_{C^0}$.) Therefore,

$$\|3uv^2 + v^3\|_{C^\alpha(B_1)} \lesssim \|u\|_{C^\alpha}\|v\|^2_{C^\alpha} + \|v\|^3_{C^\alpha} \le \|u\|_X\|v\|^2_X + \|v\|^3_X = o(\|v\|_X).$$

□

Next, we bring into play the inverse function theorem.

Theorem 3. *(Inverse function theorem for Banach spaces) Suppose that $F : X \to Y$ is a C^1 map between Banach spaces. Suppose that $x_0 \in X$ and that $dF_{x_0} : X \to Y$ is an isomorphism. Then there is a neighborhood U of x_0 so that F maps U diffeomorphically onto a neighborhood V of $F(x_0)$.*

In particular, we will use the following corollary:

Corollary 4. *Suppose that $F : X \to Y$ is a C^1 map between Banach spaces. If $dF_{x_0} : X \to Y$ is an isomorphism, then the image of F contains a neighborhood of $F(x_0)$.*

The proof of the inverse function theorem for Banach spaces is actually the same as the proof for maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. We will review/sketch the proof next lecture. In the meantime, you might see if you can remember it.

The map dN_u is an isomorphism for every $u \in X$ because of the solution to the Dirichlet problem coming from Schauder theory. We recall the statement of the result.

Theorem 5. *Suppose that*

$$Lu = \sum_{i,j} a_{ij}\partial_i\partial_j u + \sum_i b_i\partial_i u + cu,$$

where $a, b, c \in C^\alpha(B_1)$, and $c \leq 0$.
Then the map

$$u \mapsto (Lu, u|_{\partial B_1})$$

is an isomorphism from $C^{2,\alpha}(B_1)$ to $C^\alpha(B_1) \oplus C^{2,\alpha}(\partial B_1)$.

(In the case of dN_u, the 0^{th}-order coefficient c is $-3u^2 \leq 0$. And so dN_u obeys the hypotheses of Theorem 5 and so dN_u is an isomorphism.)

Corollary 6. *The image of the map $N : X \to Y$ is open.*

Proof. For each $u \in X$, dN_u is an isomorphism. By Corollary 4, the image of N contains an open neighborhood of $N(u)$. ☐

Since $N(0) = 0$, we see that the image of N contains an open neighborhood of 0. In other words,

Corollary 7. *There exists some $\epsilon > 0$, so that if $\|f\|_{C^\alpha(B_1)} < \epsilon$ and $\|\phi\|_{C^{2,\alpha}(B_1)} < \epsilon$, then there is a function $u \in C^{2,\alpha}(B_1)$ so that*

$$\triangle u - u^3 = f; u|_{\partial B_1} = \phi.$$

3. The method of continuity

If $F : X \to Y$ is a C^1 map between Banach spaces, and dF_x is an isomorphism for every $x \in X$, does it follow that F is surjective? The answer is no, even if X and Y are just the 1-dimensional Banach space $\mathbb{R}$. Consider the map $F(x) = \arctan x$. The map F is a diffeomorphism from $\mathbb{R}$ to $(-\pi/2, \pi/2)$. At each x, dF_x is an isomorphism, but the image of F is not $\mathbb{R}$.

At this point, estimates about F come into play. To illustrate the idea, we begin with maps $F : \mathbb{R}^n \to \mathbb{R}^n$.

Proposition 8. *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 map, and suppose that dF_x is an isomorphism for all $x \in \mathbb{R}^n$.*
Also suppose that $|x| \leq g(|F(x)|)$ for some continuous function $g : \mathbb{R} \to \mathbb{R}$.
Then F is surjective.

Proof. (Method of continuity). Let $y \in \mathbb{R}^n$. We want to show that there exists x with $F(x) = y$. We pick any $x_0 \in \mathbb{R}^n$, and we let $y_0 = F(x_0)$. Then we define y_t by

$$y_t = ty + (1-t)y_0.$$

We have $y_0 = F(x_0)$ and $y_1 = y$. We also note that y_t is changing continuously in t, and $|y_t|$ is uniformly bounded.

We define

$$\text{Sol} := \{t \in [0,1] | y_t \text{ lies in the image of } F\}.$$

So far, it is clear that $0 \in \text{Sol}$. The isomorphism condition will show that Sol is open. Our inequality on $|x|$ will show that Sol is closed. Therefore, we will have $\text{Sol} = [0,1]$, and so $y = y_1$ will lie in the image of F.

Let us check that Sol is open. Suppose that $t \in \text{Sol}$, and so y_t is in the image of F. Say $F(x) = y_t$. We know that dF_x is an isomorphism, and so the image of F contains a neighborhood of y_t. Therefore, there is some $\epsilon > 0$ so that the image of F contains $y_{t'}$ for all $|t' - t| < \epsilon$.

Now let us check that Sol is closed. Suppose that $t_j \in \text{Sol}$ with $t_j \to t_\infty$. Since $t_j \in \text{Sol}$, there exists x_j with $F(x_j) = y_{t_j}$. By our inequality, $|x_j| \le g(|y_{t_j}|)$ is uniformly bounded. By compactness, there is a subsequence of x_j that converges to some point x_∞. Now by continuity $F(x_\infty) = \lim F(x_j) = \lim y_{t_j} = y_{t_\infty}$. □

Based on this proof, we see that we should prove an estimate for our non-linear operator N.

Proposition 9. *If $u \in C^2(B_1)$, and if $Nu = (\triangle u - u^3, u|_{\partial B_1})$ is in $Y = C^\alpha(B_1) \oplus C^{2,\alpha}(\partial B_1)$, then*

$$\|u\|_{C^{2,\alpha}(B_1)} \le g(\|Nu\|_Y),$$

for a continuous function $g : \mathbb{R} \to \mathbb{R}$.

The proof will show that g can be taken to be a power: $g(B) = CB^p$ for some C, p. Before proving this result, we begin with a Lemma which is a version of the maximum principle for this non-linear setting.

Lemma 10. *(C^0 bound) If $u \in C^2(B_1)$, $\triangle u - u^3 = f$, and $u|_{\partial B_1} = \phi$, then*

$$\|u\|_{C^0} \lesssim \|f\|_{C^0} + \|\phi\|_{C^0}.$$

Proof. We will bound $\max_{B_1} u$. A similar argument applies to $\max_{B_1}(-u)$.

Let $w = u + Ax_1^2$ for a constant A that we will choose later, where x_1 is one of the coordinates. We study the maximum of w. Suppose that w has an interior maximum at $x_0 \in B_1$. At x_0, we must have $\nabla w(x_0) = 0$ and $\triangle w(x_0) \le 0$. Therefore,

$$0 \geq \triangle w(x_0) = \triangle u(x_0) + 2A = u(x_0)^3 + f(x_0) + 2A.$$

We choose $A = \|f\|_{C^0}$, so that $f(x_0) + 2A > 0$. We conclude that if x_0 is a local maximum of w, then $u(x_0) < 0$ and so $w(x_0) < A$. Therefore,

$$\max_{B_1} u \leq \max_{B_1} w \leq \max_{\partial B_1} w + A \leq \max_{\partial B_1} u + 2A \leq \|\phi\|_{C^0} + 2\|f\|_{C^0}.$$

$\square$

Remark. This proof depended on the signs working out in our favor. If we were studying the PDE $\triangle u + u^3 = f$, then the signs would have worked out differently, and we could have had $u(x_0) \gg 0$ at a maximum of w. The maximum principle would then fail, and we would get no good bounds. Indeed, for this equation, there is no good regularity estimate of the form in Proposition 9.

Proof of Proposition 9. Suppose $u \in C^2(B_1)$, $\triangle u - u^3 = f$, and $u|_{\partial B_1} = \phi$. We want to show that $\|u\|_{C^{2,\alpha}} \leq g(\|f\|_{C^\alpha} + \|\phi\|_{C^{2,\alpha}})$.

We apply the global Schauder inequality.

$$\|u\|_{C^{2,\alpha}(B_1)} \lesssim \|\triangle u\|_{C^\alpha} + \|\phi\|_{C^{2,\alpha}} \leq \|u^3\|_{C^\alpha} + \|f\|_{C^\alpha} + \|\phi\|_{C^{2,\alpha}}.$$

We have to deal with the term $\|u^3\|_{C^\alpha}$ on the right hand side. In general, $\|fg\|_{C^\alpha} \leq \|f\|_{C^\alpha}\|g\|_{C^0} + \|f\|_{C^0}\|g\|_{C^\alpha}$, and so we have

$$\|u^3\|_{C^\alpha} \leq 10\|u\|_{C^0}^2\|u\|_{C^\alpha}.$$

Applying the C^0 bound from Lemma 10, we get

$$\|u^3\|_{C^\alpha} \lesssim (\|f\|_{C^0} + \|\phi\|_{C^0})^2\|u\|_{C^\alpha}.$$

Now we can use the Peter-Paul type inequality

$$\|u\|_{C^\alpha(B_1)} \leq \epsilon\|u\|_{C^{2,\alpha}(B_1)} + C\epsilon^{-s}\|u\|_{C^0(B_1)},$$

for some exponent $s = s(\alpha)$. We choose ϵ small enough so that $(\|f\|_{C^0}+\|\phi\|_{C^0})^2\epsilon \ll 1$. Plugging this into our original inequality, we get:

$$\|u\|_{C^{2,\alpha}(B_1)} \leq (1/2)\|u\|_{C^{2,\alpha}(B_1)} + C(\|f\|_{C^0} + \|\phi\|_{C^0})^{2s}\|u\|_{C^0} + C\|f\|_{C^\alpha} + C\|\phi\|_{C^{2,\alpha}}.$$

Now we can rearrange the dangerous term $(1/2)\|u\|_{C^{2,\alpha}(B_1)}$, and use Lemma 10 to control $\|u\|_{C^0}$, and we get the desired estimate. $\square$

Using the method of continuity and Proposition 9, we will finish the proof of Theorem 1 next class.