Lecture Notes for LG's Diff. Analysis

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DiGeorgi-Nash-Moser Theorem

1 Classical Approach

Our goal in these notes will be to prove the following theorem:

Theorem 1.1 (DiGeorgi-Nash-Moser)**.** Let

$$Lu := \sum \partial_i(a_{ij}\partial_j u) \text{ and } 0 < \lambda \leq a_{ij} \leq \Lambda \tag{DGH}$$

Then there exists $\alpha(n, \lambda, \Lambda) > 0$ and $C(n, \lambda, \Lambda)$ such that if $Lu = 0$, then

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(\lambda, \Lambda, n)\|u\|_{C^0(B_1)}$$

Note that this estimate does not in any way involve derivatives of the a_{ij}.

We start by reminding of the Dirichlet energy of a function:

Definition 1.1 (Dirichlet Energy)**.** If $u : \Omega \to \mathbb{R}$, then $E(u) = \int_\Omega |\nabla u|^2$.

With this, we have the following easy proposition.

Proposition 1.1. If $u, w \in C^2(\bar{\Omega})$, $u = w$ on $\partial\Omega$, and $\Delta u = 0$, then $E(u) \leq E(w)$.

Proof. : Let $w = u + v$, so $v|_{\partial\Omega} = 0$. Then

$$\begin{aligned} E(w) &= \int_\Omega \langle \nabla w, \nabla w \rangle = \int_\Omega |\nabla u|^2 + |\nabla v|^2 + 2\int_\Omega \nabla u \cdot \nabla v \\ &\leq \int_\Omega |\nabla u|^2 = E(u) \end{aligned}$$

where we got from the first line to the second by integration by parts. □

In a similar way, we can define

Definition 1.2 (Gen. Dirichlet Energy)**.** If L, a satisfies (DGH), then

$$E_a(u) = \int_\Omega \sum a_{ij}(\partial_i u)(\partial_j u)$$

and get a similar proposition with identical proof:

Proposition 1.2. If $u, w \in C^2(\bar{\Omega})$, and $u = w$ on $\partial\Omega$, and $Lu = 0$, then $E_a(w) \geq E_a(u)$.

We now prove an L^2 estimate relating ∇u to u.

Proposition 1.3. If L follows (DGH) and $Lu = 0$ on B_1 then

$$\int_{B_{1/2}} |\nabla u|^2 \lesssim \int_{B_1} |u|^2$$

Proof. We will use integration by parts and localization. Let $\eta = 1$ on $B_{1/2}$ and be 0 outside of B_1.

$$\begin{aligned} \int_{B_{1/2}} |\nabla u|^2 &\leq \int \eta^2 |\nabla u|^2 \approx \int \eta^2 \sum a_{ij} \partial_i u \partial_j u \\ &\leq \int \eta^2 (Lu) u + \int |\nabla \eta| \eta |\nabla u| |u| \\ &\leq \left(\int \eta^2 |\nabla u|^2 \right)^{1/2} \left(\int |\nabla \eta|^2 u^2 \right)^{1/2} \end{aligned}$$

□

A classical approach would be to then prove the following:

Proposition 1.4. If (DGH), $Lu = 0$ and $\|a_{ij}\|_{C^1} \leq B$ then

$$\int_{B_{1/2}} |D^2 u|^2 \leq C(B, n, \lambda, \Lambda) \int_{B_{3/4}} |\nabla u|^2$$

Proof. We have that $0 = \partial_k Lu = L(\partial_k u) + (\partial_k a_{ij})\partial_i \partial_j u$. Then,

$$\begin{aligned}\int_{B_{1/2}} |D^2 u|^2 &\lesssim \int \eta^2 \sum a_{ij} \partial_i \partial_k u \partial_j \partial_k u \\ &\lesssim \int |\nabla \eta| \eta |D^2 u| |\nabla u| + \int \eta^2 L(\partial_k u) \partial_k u \\ &\lesssim \int |\nabla \eta| \eta |D^2 u| |\nabla u| + \int \eta^2 B |D^2 u| |\nabla u|\end{aligned}$$

The result comes from applying Cauchy-Schwartz to this last pair of terms. □

However, this won't get us closer to proving DiGeorgi-Nash-Moser because we're using an estimate on the derivatives of a in our inequality. Looks like we'll have to be clever!

2 L^∞ Bound

Theorem 2.1 (DGNM L^∞ bound)**.** Let L satisfy (DGH), $Lu \geq 0$, $u > 0$. Then

$$\|u\|_{L^\infty(B_{1/2})} \leq \|u\|_{L^2(B_1)}$$

Proof. We start with a lemma:

Lemma 2.1. Under the hypotheses, and if $1/2 \leq r < r + w \leq 1$ then

$$\|\nabla u\|_{L^2(B_r)} \lesssim \|u\|_{L^2(B_{r+w})} w^{-1}$$

Proof. Let $\eta = 1$ on B_r and 0 on B^c_{r+w}. Note that η can be constructed so that $|\nabla\eta| < 2w^{-1}$. Then the proof proceeds in exactly the same fashion as Proposition 1.3. $\square$

Lemma 2.2. Under hypotheses, and $1/2 \leq r < r + 2 \leq 1$, we have

$$\|u\|_{L^{2n/(n-2)}(B_r)} \lesssim w^{-1}\|u\|_{L^2(B_{r+w})}$$

Proof. Consider ηu with $\eta = 1$ on B_r, and 0 outside of $B_{r+w/2}$. Then by the Sobolev inequality, we have

$$\begin{aligned} \|\eta u\|_{L^{2n/(n-2)}} &\lesssim \|\nabla(\eta u)\|_{L^2} \\ &\leq \|(\nabla\eta)u\|_{L^2} + \|\eta(\nabla u)\|_{L^2} \end{aligned}$$

Also, we have that

$$\begin{aligned} \|(\nabla\eta)u\|_{L^2} &\leq \|\nabla\eta\|_\infty\|u\|_{L^2(B_{r+w/2})} \lesssim w^{-1}\|u\|_{L^2(B_{r+w})} \\ \|\eta(\nabla u)\|_{L^2} &\leq \|\nabla u\|_{L^2(B_{r+w/2})} \lesssim w^{-1}\|u\|_{L^2(B_{r+w})} \end{aligned}$$

$\square$

Lemma 2.3. If $\beta > 1$, $Lu \geq 0$ and $u > 0$, then $Lu^\beta \geq 0$.

Proof. Compute:

$$\begin{aligned} Lu^\beta &= \sum \partial_i(a_{ij}\partial_j(u^\beta)) = \sum \partial_i(a_{ij}\beta u^{\beta-1}\partial_j u) \\ &= (Lu)(\beta u^{\beta-1}) + \sum a_{ij}\partial_i u\partial_j u\beta(\beta-1)u^{\beta-2} \geq 0 \end{aligned}$$

where the last inequality comes from ellipticity of a_{ij}. $\square$

Now, apply Lemma 2.2 to u^β to get

$$\|u^\beta\|_{L^{2n/(n-2)}(B_r)} \lesssim w^{-1}\|u^\beta\|_{L^2(B_{r+w})}$$

Rewriting this with $s = \frac{n}{n-2}$ we get

Lemma 2.4. If $1/2 \le r < r + w \le 1$ and $p \ge 2$, then

$$\|u\|_{L^{sp}(B_r)} \le (Cw^{-1})^{2/p} \|u\|_{L^p(B_{r+w})}$$

For the next step, we iterate this lemma. If we have $1 = r_0 > r_1 > \cdots > r_k > 1/2$, then we get the sequence of inequalities

$$\|u\|_{L^2(B_1)} \ge A_0 \|u\|_{L^{2s}(B_{r_1})} \ge \cdots \ge A_0 \cdots A_{k-1} \|u\|_{L^{2s^k}(B_{r_k})}$$

where the A_j are given by Lemma 2.4. Let's pick $r_j = \frac{1}{2} + \frac{1}{j+2}$, so that $r_j - r_{j+1} \approx j^{-2}$. Thus, $A_j = (C(r_j - r_{j-1})^{-1})^{s^{-j}}$. Therefore,

$$\begin{aligned}\log(\prod A_j) \le \sum \log(A_j) &\le \sum_{j=0}^{\infty} s^{-j}(C + C\log(r_j - r_{j+1})) \\ &\le \sum_{j=0}^{\infty} s^{-j}(C + C\log j) < \infty\end{aligned}$$

□

3 Finishing the Proof

Recall the Harnack inequality:

Theorem 3.1 (Harnack)**.** If $\Delta u = 0$ on B_1 and $u > 0$ then $\min_{B_{1/2}} u \ge \gamma(n) \max_{B_1} u$.

We will show a Harnack inequality for our L which satisfies (DGH).

Theorem 3.2 (DGNM Harnack)**.** If L satisfies (DGH), $Lu = 0$, $1 > u > 0$ on B_1, and

$$|\{x \in B_{1/2} | u(x) > 1/10\}| \ge \frac{1}{10}|B_{1/2}| \tag{P}$$

then $\min_{B_{1/2}} u \ge \gamma(n)$.

For now, let's assume this theorem, and see how it implies the DiGeorgi-Nash-Moser estimate.

Definition 3.1. $\operatorname{osc}_\Omega u := \sup_\Omega u - \inf_\Omega u$.

Corollary 3.1. If $Lu = 0$ on Ω, $B_r(x) \subset \Omega$, then

$$\operatorname{osc}_{B_{r/2}(x)} u \leq (1-\gamma) \operatorname{osc}_{B_r(x)} u \tag{O}$$

Proof. We start with some simple reductions via scaling. Without loss of generality, we can take:

$$\inf_{B_r(x)} u = 0, \ \sup_{B_r(x)} u = 1, \ r = 1$$
$$|\{x \in B_{1/2} | u(x) \geq 1/2\}| \geq B_{1/2}/2$$

Thus by DGNM Harnack, $\min_{B_{1/2}} u \geq \gamma$, and thus $\operatorname{osc}_{B_{1/2}} u \leq 1 - \gamma = (1-\gamma)\operatorname{osc}_{B_1} u$ □

Now we can complete the proof with the following:

Proposition 3.1. Let $u : B_1 \to \mathbb{R}$ satisfy (O). Then $\|u\|_{C^\alpha(B_{1/2})} \lesssim \|u\|_{C^0(B_1)}$ for some $\alpha = \alpha(\gamma) > 0$.

Proof. Let $x, y \in B^{1/2}$, $|x - y| = d$ and $a = (x+y)/2$. Then

$$|u(x) - u(y)| \leq (\operatorname{osc}_{B_d(a)} u)(1-\gamma) \leq \cdots \leq (1-\gamma)^k \operatorname{osc}_{B_{2^k d}(a)} u$$

Choose k such that $1/4 < 2^k d \leq 1/2$. Then $k = \log_2(1/d) + O(1)$, and so

$$|u(x) - u(y)| \leq (1-\gamma)^k \operatorname{osc}_{B_1} u \leq 2(1-\gamma)^k |u\|_{C^0(B_1)}.$$

Also,

$$(1-\gamma)^k \leq 4(1-\gamma)^{\log_2(1/d)} = 4d^{-\log_2(1-\gamma)}.$$

Therefore, setting $\alpha(\gamma) = -\log_2(1-\gamma) \approx \gamma + O(\gamma^2)$, we get our proposition. □

Now let's prove the Harnack inequality. Before we do the DGNM Harnack, we'll remember how the normal Δ Harnack inequality works:

Lemma 3.1. If $\Delta u = 0$ and $u > 0$ then $\|\nabla \log u\|_{L^\infty(B_{1/2})} \lesssim 1$.

Note that the lemma implies the Harnack inequality by integrating.

Proof. We have $\nabla \log u = \frac{\nabla u}{u}$. Also, by elliptic regularity, we have that

$$|\nabla u|(x) \lesssim \|u\|_{L^1(B_{1/2}(x))} = \int_{B_{1/2}(x)} u = |B_{1/2}(x)| u(x)$$

so that $|\nabla u|/u \lesssim 1$. □

With this method in mind, let's prove the DGNM Harnack.

DGNM Harnack.

Lemma 3.2. If L satisfies (DGH), $Lu = 0$, $u > 0$ on B_1 then $\|\nabla \log u\|_{L^2(B_{1/2})} \lesssim 1$.

Proof. Pick a nice cutoff function η as usual.

$$\begin{aligned}
\int_{B_{1/2}} |\nabla \log u|^2 &= \int \eta^2 |\nabla \log u|^2 \lesssim \int \eta^2 \sum a_{ij} \partial_i \log u \partial_j \log u \\
&= \int \eta^2 \sum a_{ij} \frac{\partial_i u}{u} \frac{\partial_j u}{u} = - \int \eta^2 \sum a_{ij} \partial_i u \partial_j u^{-1} \\
&\lesssim \int \eta |\nabla \eta| |\nabla u| u^{-1} = \int \eta |\nabla \eta| |\nabla \log u| \\
&\leq \left(\int \eta^2 |\nabla \log u|^2 \right)^{1/2} \left(\int |\nabla \eta|^2 \right)^{1/2}
\end{aligned}$$

□

Letting $w = -\log u$, we have that $\|\nabla w\|_{L^2(B_{9/10})} \lesssim 1$. We want an L^∞ bound on w. By (P), we have that

$$|\{x \in B_{1/2} | w \leq \log 10\}| \geq \frac{1}{10} |B_{1/2}|$$

Now we use the Poincare Inequality:

Theorem 3.3 (Poincare)**.** If (P) then $\int_{B_{8/10}} |w|^2 \lesssim \int_{B_{9/10}} |\nabla w|^2 + 1$

Therefore, we have an L^2 bound on w instead of ∇w. Now we have

Lemma 3.3. $Lw \geq 0$

Proof. Compute:

$$\begin{aligned} -\sum \partial_i(a_{ij}\partial_j \log u) &= -\sum \partial_i(a_{ij}(\partial_j u)u^{-1}) \\ &= Lu \cdot u^{-1} + \sum a_{ij}(\partial_i u)(\partial_j u)u^{-2} \geq 0 \end{aligned}$$

□

Finally, $w = -\log u > 0$ because $u < 1$, and so we can apply Theorem 2.1 and get

$$\|w\|_{L^\infty(B_{1/2})} \lesssim \|w\|_{L^2(B_{8/10})} \lesssim 1$$

thus completing the proof of the Harnack inequality.

□