4. Integration

The (μ)-*integral* of a non-negative simple function is by definition

$$\int_Y f\,d\mu = \sum_i a_i \mu(Y \cap E_i)\,,\ Y \in \mathcal{M}\,. \tag{4.1}$$

Here the convention is that if $\mu(Y \cap E_i) = \infty$ but $a_i = 0$ then $a_i \cdot \mu(Y \cap E_i) = 0$. Clearly this integral takes values in $[0, \infty]$. More significantly, if $c \geq 0$ is a constant and f and g are two non-negative (μ-measurable) simple functions then

$$\begin{gathered}\int_Y cf d\mu = c\int_Y f d\mu \\ \int_Y (f+g)d\mu = \int_Y f d\mu + \int_Y g d\mu \\ 0 \leq f \leq g \ \Rightarrow \ \int_Y f\,d\mu \leq \int_Y g\,d\mu\,.\end{gathered} \tag{4.2}$$

(See [1] Proposition 2.13 on page 48.)

To see this, observe that (4.1) holds for *any* presentation (3.5) of f with all $a_i \geq 0$. Indeed, by restriction to E_i and division by a_i (which can be assumed non-zero) it is enough to consider the special case

$$\chi_E = \sum_j b_j \chi_{F_j}.$$

The F_j can always be written as the union of a finite number, N', of disjoint measurable sets, $F_j = \cup_{l \in S_j} G_l$ where $j = 1, \dots, N$ and $S_j \subset \{1, \dots, N'\}$. Thus

$$\sum_j b_j \mu(F_j) = \sum_j b_j \sum_{l \in S_j} \mu(G_l) = \mu(E)$$

since $\sum_{\{j; l \in S_j\}} b_j = 1$ for each j.

From this all the statements follow easily.

Definition 4.1. *For a non-negative μ-measurable extended function $f : X \longrightarrow [0, \infty]$ the integral (with respect to μ) over any measurable set $E \subset X$ is*

$$\int_E f d\mu = \sup\{\int_E h d\mu;\ 0 \leq h \leq f,\ h \textit{ simple and measurable.}\} \tag{4.3}$$

By taking suprema, $\int_E f d\mu$ has the first and last properties in (4.2). It also has the middle property, but this is less obvious. To see this, we shall prove the basic 'Monotone convergence theorem' (of Lebesgue). Before doing so however, note what the vanishing of the integral means.

Lemma 4.2. *If $f : X \longrightarrow [0, \infty]$ is measurable then $\int_E f d\mu = 0$ for a measurable set E if and only if*

$$\{x \in E; f(x) > 0\} \text{ has measure zero.} \tag{4.4}$$

Proof. If (4.4) holds, then any positive simple function bounded above by f must also vanish outside a set of measure zero, so its integral must be zero and hence $\int_E f d\mu = 0$. Conversely, observe that the set in (4.4) can be written as

$$E_n = \bigcup_n \{x \in E; f(x) > 1/n\}.$$

Since these sets increase with n, if (4.4) does not hold then one of these must have positive measure. In that case the simple function $n^{-1}\chi_{E_n}$ has positive integral so $\int_E f d\mu > 0$. □

Notice the fundamental difference in approach here between Riemann and Lebesgue integrals. The Lebesgue integral, (4.3), uses approximation by functions constant on possibly quite nasty measurable sets, not just intervals as in the Riemann lower and upper integrals.

Theorem 4.3 (Monotone Convergence)**.** *Let f_n be an increasing sequence of non-negative measurable (extended) functions, then $f(x) = \lim_{n\to\infty} f_n(x)$ is measurable and*

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu \tag{4.5}$$

for any measurable set $E \subset X$.

Proof. To see that f is measurable, observe that

$$f^{-1}(a, \infty] = \bigcup_n f_n^{-1}(a, \infty]. \tag{4.6}$$

Since the sets $(a, \infty]$ generate the Borel σ-algebra this shows that f is measurable.

So we proceed to prove the main part of the proposition, which is (4.5). Rudin has quite a nice proof of this, [5] page 21. Here I paraphrase it. We can easily see from (4.1) that

$$\alpha = \sup \int_E f_n d\mu = \lim_{n\to\infty} \int_E f_n d\mu \le \int_E f d\mu.$$

Given a simple measurable function g with $0 \le g \le f$ and $0 < c < 1$ consider the sets $E_n = \{x \in E; f_n(x) \ge cg(x)\}$. These are measurable and increase with n. Moreover $E = \bigcup_n E_n$. It follows that

$$\int_E f_n d\mu \ge \int_{E_n} f_n d\mu \ge c \int_{E_n} g d\mu = \sum_i a_i \mu(E_n \cap F_i) \tag{4.7}$$

in terms of the natural presentation of $g = \sum_i a_i \chi_{F_i}$. Now, the fact that the E_n are measurable and increase to E shows that

$$\mu(E_n \cap F_i) \to \mu(E \cap F_i)$$

as $n \to \infty$. Thus the right side of (4.7) tends to $c \int_E g d\mu$ as $n \to \infty$. Hence $\alpha \geq c \int_E g d\mu$ for all $0 < c < 1$. Taking the supremum over c and then over all such g shows that

$$\alpha = \lim_{n \to \infty} \int_E f_n d\mu \geq \sup \int_E g d\mu = \int_E f d\mu.$$

They must therefore be equal. □

Now for instance the additivity in (4.1) for $f \geq 0$ and $g \geq 0$ any measurable functions follows from Proposition 3.3. Thus if $f \geq 0$ is measurable and f_n is an approximating sequence as in the Proposition then $\int_E f d\mu = \lim_{n \to \infty} \int_E f_n d\mu$. So if f and g are two non-negative measurable functions then $f_n(x) + g_n(x) \uparrow f + g(x)$ which shows not only that $f + g$ is measurable by also that

$$\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu.$$

As with the definition of u_+ long ago, this allows us to extend the definition of the integral to any *integrable* function.

Definition 4.4. *A measurable extended function $f : X \longrightarrow [-\infty, \infty]$ is said to be integrable on E if its positive and negative parts both have finite integrals over E, and then*

$$\int_E f d\mu = \int_E f_+ d\mu - \int_E f_- d\mu.$$

Notice if f is μ-integrable then so is $|f|$. One of the objects we wish to study is the space of integrable functions. The fact that the integral of $|f|$ can vanish encourages us to look at what at first seems a much more complicated object. Namely we consider an equivalence relation between integrable functions

$$f_1 \equiv f_2 \Longleftrightarrow \mu(\{x \in X; f_1(x) \neq f_2(x)\}) = 0. \tag{4.8}$$

That is we identify two such functions if they are equal 'off a set of measure zero.' Clearly if $f_1 \equiv f_2$ in this sense then

$$\int_X |f_1| d\mu = \int_X |f_2| d\mu = 0, \ \int_X f_1 d\mu = \int_X f_2 d\mu.$$

A necessary condition for a measurable function $f \geq 0$ to be integrable is

$$\mu\{x \in X; f(x) = \infty\} = 0.$$

Let E be the (necessarily measureable) set where $f = \infty$. Indeed, if this does not have measure zero, then the sequence of simple functions $n\chi_E \le f$ has integral tending to infinity. It follows that each equivalence class under (4.8) has a representative which is an honest function, i.e. which is finite everywhere. Namely if f is one representative then

$$f'(x) = \begin{cases} f(x) & x \notin E \\ 0 & x \in E \end{cases}$$

is also a representative.

We shall denote by $L^1(X, \mu)$ the space consisting of such equivalence classes of integrable functions. This is a normed linear space as I ask you to show in Problem 11.

The monotone convergence theorem often occurrs in the slightly disguised form of Fatou's Lemma.

Lemma 4.5 (Fatou)**.** *If f_k is a sequence of non-negative integrable functions then*

$$\int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu \, .$$

Proof. Set $F_k(x) = \inf_{n\ge k} f_n(x)$. Thus F_k is an increasing sequence of non-negative functions with limiting function $\liminf_{n\to\infty} f_n$ and $F_k(x) \le f_n(x) \, \forall \; n \ge k$. By the monotone convergence theorem

$$\int \liminf_{n\to\infty} f_n \, d\mu = \lim_{k\to\infty} \int F_k(x) \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu.$$

□

We further extend the integral to complex-valued functions, just saying that

$$f : X \to \mathbb{C}$$

is integrable if its real and imaginary parts are both integrable. Then, by definition,

$$\int_E f d\mu = \int_E \operatorname{Re} f d\mu + i \int_E \operatorname{Im} f d\mu$$

for any $E \subset X$ measurable. It follows that if f is integrable then so is $|f|$. Furthermore

$$\left| \int_E f \, d\mu \right| \le \int_E |f| \; d\mu \, .$$

This is obvious if $\int_E f \, d\mu = 0$, and if not then

$$\int_E f \, d\mu = Re^{i\theta} \; R > 0 \, , \; \theta \subset [0, 2\pi) \, .$$

Then

$$\begin{aligned}\left|\int_E f\,d\mu\right| &= e^{-i\theta}\int_E f\,d\mu\\ &= \int_E e^{-i\theta} f\,d\mu\\ &= \int_E \mathbb{R}e(e^{-i\theta} f)\,d\mu\\ &\le \int_E \left|\mathbb{R}e(e^{-i\theta} f)\right|\,d\mu\\ &\le \int_E \left|e^{-i\theta} f\right|\,d\mu = \int_E |f|\,d\mu\,.\end{aligned}$$

The other important convergence result for integrals is Lebesgue's *Dominated convergence theorem.*

Theorem 4.6. *If f_n is a sequence of integrable functions, $f_k \to f$ a.e.*[5] *and $|f_n| \le g$ for some integrable g then f is integrable and*

$$\int f d\mu = \lim_{n\to\infty}\int f_n d\mu\,.$$

Proof. First we can make the sequence $f_n(x)$ converge by changing all the $f_n(x)$'s to zero on a set of measure zero outside which they converge. This does not change the conclusions. Moreover, it suffices to suppose that the f_n are real-valued. Then consider

$$h_k = g - f_k \ge 0\,.$$

Now, $\liminf_{k\to\infty} h_k = g - f$ by the convergence of f_n; in particular f is integrable. By monotone convergence and Fatou's lemma

$$\begin{aligned}\int (g-f)d\mu = \int \liminf_{k\to\infty} h_k\,d\mu &\le \liminf_{k\to\infty}\int (g-f_k)\,d\mu\\ &= \int g\,d\mu - \limsup_{k\to\infty}\int f_k\,d\mu\,.\end{aligned}$$

Similarly, if $H_k = g + f_k$ then

$$\int (g+f)d\mu = \int \liminf_{k\to\infty} H_k\,d\mu \le \int g\,d\mu + \liminf_{k\to\infty}\int f_k\,d\mu.$$

It follows that

$$\limsup_{k\to\infty}\int f_k\,d\mu \le \int f\,d\mu \le \liminf_{k\to\infty}\int f_k\,d\mu.$$

[5]Means on the complement of a set of measure zero.

Thus in fact

$$\int f_k \, d\mu \to \int f \, d\mu \, .$$

□

Having proved Lebesgue's theorem of dominated convergence, let me use it to show something important. As before, let μ be a positive measure on X. We have defined $L^1(X, \mu)$; let me consider the more general space $L^p(X, \mu)$. A measurable function

$$f : X \to \mathbb{C}$$

is said to be 'L^p', for $1 \le p < \infty$, if $|f|^p$ is integrable[6], i.e.,

$$\int_X |f|^p \, d\mu < \infty \, .$$

As before we consider equivalence classes of such functions under the equivalence relation

$$f \sim g \Leftrightarrow \mu \left\{ x ; (f-g)(x) \neq 0 \right\} = 0 \, . \tag{4.9}$$

We denote by $L^p(X, \mu)$ the space of such equivalence classes. It is a linear space and the function

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p} \tag{4.10}$$

is a norm (we always assume $1 \le p < \infty$, sometimes $p = 1$ is excluded but later $p = \infty$ is allowed). It is straightforward to check everything except the triangle inequality. For this we start with

Lemma 4.7. *If $a \ge 0$, $b \ge 0$ and $0 < \gamma < 1$ then*

$$a^\gamma b^{1-\gamma} \le \gamma a + (1-\gamma) b \tag{4.11}$$

with equality only when $a = b$.

Proof. If $b = 0$ this is easy. So assume $b > 0$ and divide by b. Taking $t = a/b$ we must show

$$t^\gamma \le \gamma t + 1 - \gamma \, , \; 0 \le t \, , \; 0 < \gamma < 1 \, . \tag{4.12}$$

The function $f(t) = t^\gamma - \gamma t$ is differentiable for $t > 0$ with derivative $\gamma t^{\gamma - 1} - \gamma$, which is positive for $t < 1$ and negative for $t > 1$. Thus $f(t) \le f(1)$ with equality only for $t = 1$. Since $f(1) = 1 - \gamma$, this is (4.12), proving the lemma. □

We use this to prove Hölder's inequality

[6] Check that $|f|^p$ is automatically measurable.

Lemma 4.8. *If f and g are measurable then*

$$\left|\int fg d\mu\right| \leq \|f\|_p \|g\|_q \tag{4.13}$$

for any $1 < p < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. If $\|f\|_p = 0$ or $\|g\|_q = 0$ the result is trivial, as it is if either is infinite. Thus consider

$$a = \left|\frac{f(x)}{\|f\|_p}\right|^p , \; b = \left|\frac{g(x)}{\|g\|_q}\right|^q$$

and apply (4.11) with $\gamma = \frac{1}{p}$. This gives

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q} .$$

Integrating over X we find

$$\begin{aligned}\frac{1}{\|f\|_p\|g\|_q} \int_X |f(x)g(x)| \; d\mu \\ \leq \frac{1}{p} + \frac{1}{q} = 1 .\end{aligned}$$

Since $\left|\int_X fg \, d\mu\right| \leq \int_X |fg| \; d\mu$ this implies (4.13).

□

The final inequality we need is *Minkowski's* inequality.

Proposition 4.9. *If $1 < p < \infty$ and $f, g \in L^p(X, \mu)$ then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p . \tag{4.14}$$

Proof. The case $p = 1$ you have already done. It is also obvious if $f + g = 0$ a.e.. If not we can write

$$|f + g|^p \leq (|f| + |g|) \, |f + g|^{p-1}$$

and apply Hölder's inequality, to the right side, expanded out,

$$\int |f + g|^p \; d\mu \leq (\|f\|_p + \|g\|_p) \; , \left(\int |f + g|^{q(p-1)} \; d\mu\right)^{1/q} .$$

Since $q(p - 1) = p$ and $1 - \frac{1}{q} = 1/p$ this is just (4.14). □

So, now we know that $L^p(X, \mu)$ is a normed space for $1 \leq p < \infty$. In particular it is a metric space. One important additional property that a metric space may have is *completeness*, meaning that every Cauchy sequence is convergent.

Definition 4.10. *A normed space in which the underlying metric space is complete is called a* Banach *space.*

Theorem 4.11. *For any measure space (X, M, μ) the spaces $L^p(X, \mu)$, $1 \le p < \infty$, are Banach spaces.*

Proof. We need to show that a given Cauchy sequence $\{f_n\}$ converges in $L^p(X, \mu)$. It suffices to show that it has a convergent subsequence. By the Cauchy property, for each k $\exists$ $n = n(k)$ s.t.

$$\|f_n - f_\ell\|_p \le 2^{-k} \ \forall\ \ell \ge n\,. \tag{4.15}$$

Consider the sequence

$$g_1 = f_1\,,\ g_k = f_{n(k)} - f_{n(k-1)}\,,\ k > 1\,.$$

By (4.15), $\|g_k\|_p \le 2^{-k}$, for $k > 1$, so the series $\sum_k \|g_k\|_p$ converges, say to $B < \infty$. Now set

$$h_n(x) = \sum_{k=1}^{n} |g_k(x)|\ ,\ n \ge 1\,,\ h(x) = \sum_{k=1}^{\infty} g_k(x).$$

Then by the monotone convergence theorem

$$\int_X h^p\, d\mu = \lim_{n\to\infty} \int_X |h_n|^p\, d\mu \le B^p\,,$$

where we have also used Minkowski's inequality. Thus $h \in L^p(X, \mu)$, so the series

$$f(x) = \sum_{k=1}^{\infty} g_k(x)$$

converges (absolutely) almost everywhere. Since

$$|f(x)|^p = \lim_{n\to\infty} \left| \sum_{k=1}^{n} g_k \right|^p \le h^p$$

with $h^p \in L'(X, \mu)$, the dominated convergence theorem applies and shows that $f \in L^p(X, \mu)$. Furthermore,

$$\sum_{k=1}^{\ell} g_k(x) = f_{n(\ell)}(x) \text{ and } \left|f(x) - f_{n(\ell)}(x)\right|^p \le (2h(x))^p$$

so again by the dominated convergence theorem,

$$\int_X \left|f(x) - f_{n(\ell)}(x)\right|^p \to 0\,.$$

Thus the subsequence $f_{n(\ell)} \to f$ in $L^p(X, \mu)$, proving its completeness. $\square$

Next I want to return to our starting point and discuss the Riesz representation theorem. There are two important results in measure theory that I have not covered — I will get you to do most of them in the problems — namely the Hahn decomposition theorem and the Radon-Nikodym theorem. For the moment we can do without the latter, but I will use the former.

So, consider a locally compact metric space, X. By a Borel measure on X, or a signed Borel measure, we shall mean a function on Borel sets

$$\mu : \mathcal{B}(X) \to \mathbb{R}$$

which is given as the difference of two finite positive Borel measures

$$\mu(E) = \mu_1(E) - \mu_2(E)\,. \tag{4.16}$$

Similarly we shall say that μ is Radon, or a signed Radon measure, if it *can be written* as such a difference, with both μ_1 and μ_2 finite Radon measures. See the problems below for a discussion of this point.

Let $M_{\text{fin}}(X)$ denote the set of finite Radon measures on X. This is a normed space with

$$\|\mu\|_1 = \inf(\mu_1(X) + \mu_2(X)) \tag{4.17}$$

with the infimum over all Radon decompositions (4.16). Each signed Radon measure defines a continuous linear functional on $\mathcal{C}_0(X)$:

$$\int \cdot \, d\mu : \mathcal{C}_0(X) \ni f \longmapsto \int_X f \cdot d\mu\,. \tag{4.18}$$

Theorem 4.12 (Riesz representation.)**.** *If X is a locally compact metric space then every continuous linear functional on $\mathcal{C}_0(X)$ is given by a unique finite Radon measure on X through* (4.18)*.*

Thus the dual space of $\mathcal{C}_0(X)$ *is* $M_{\text{fin}}(X)$ – at least this is how such a result is usually interpreted

$$(\mathcal{C}_0(X))' = M_{\text{fin}}(X), \tag{4.19}$$

see the remarks following the proof.

Proof. We have done half of this already. Let me remind you of the steps.

We started with $u \in (\mathcal{C}_0(X))'$ and showed that $u = u_+ - u_-$ where $u_\pm$ are *positive* continuous linear functionals; this is Lemma 1.5. Then we showed that $u \geq 0$ defines a finite positive Radon measure μ. Here μ is defined by (1.11) on open sets and $\mu(E) = \mu^*(E)$ is given by (1.12) on general Borel sets. It is finite because

$$\begin{aligned}\mu(X) = \sup\{u(f)\,;\, 0 \le f \le 1\,,\, \operatorname{supp} f \Subset X\,,\, f \in C(X)\} \\ \le \|u\|\,. \end{aligned} \tag{4.20}$$

From Proposition 2.8 we conclude that μ is a Radon measure. Since this argument applies to $u_\pm$ we get two positive finite Radon measures $\mu_\pm$ and hence a signed Radon measure

$$\mu = \mu_+ - \mu_- \in M_{\text{fin}}(X). \tag{4.21}$$

In the problems you are supposed to prove the Hahn decomposition theorem, in particular in Problem 14 I ask you to show that (4.21) *is* the Hahn decomposition of μ — this means that there is a Borel set $E \subset X$ such that $\mu_-(E) = 0\,,\, \mu_+(X \setminus E) = 0$.

What we have defined is a linear map

$$(\mathcal{C}_0(X))' \to M(X),\ u \longmapsto \mu\,. \tag{4.22}$$

We want to show that this is an isomorphism, i.e., it is $1-1$ and onto.

We first show that it is $1-1$. That is, suppose $\mu = 0$. Given the uniqueness of the Hahn decomposition this implies that $\mu_+ = \mu_- = 0$. So we can suppose that $u \ge 0$ and $\mu = \mu_+ = 0$ and we have to show that $u = 0$; this is obvious since

$$\begin{aligned}\mu(X) = \sup\{u(f);\ \operatorname{supp} u \Subset X,\ 0 \le f \le 1\ f \in C(X)\} = 0 \\ \Rightarrow u(f) = 0 \text{ for all such } f\,. \end{aligned} \tag{4.23}$$

If $0 \le f \in C(X)$ and $\operatorname{supp} f \Subset X$ then $f' = f/\|f\|_\infty$ is of this type so $u(f) = 0$ for every $0 \le f \in C(X)$ of compact support. From the decomposition of continuous functions into positive and negative parts it follows that $u(f) = 0$ for every f of compact support. Now, if $f \in \mathcal{C}_o(X)$, then given $n \in \mathbb{N}$ there exists $K \Subset X$ such that $|f| < 1/n$ on $X \setminus K$. As you showed in the problems, there exists $\chi \in \mathcal{C}(X)$ with $\operatorname{supp}(\chi) \Subset X$ and $\chi = 1$ on K. Thus if $f_n = \chi f$ then $\operatorname{supp}(f_n) \Subset X$ and $\|f - f_n\| = \sup(|f - f_n| < 1/n$. This shows that $\mathcal{C}_0(X)$ is the closure of the subspace of continuous functions of compact support so by the assumed *continuity* of u, $u = 0$.

So it remains to show that *every* finite Radon measure on X arises from (4.22). We do this by starting from μ and constructing u. Again we use the Hahn decomposition of μ, as in (4.21)[7]. Thus we assume $\mu \ge 0$ and construct u. It is obvious what we want, namely

$$u(f) = \int_X f\, d\mu\,,\ f \in \mathcal{C}_c(X)\,. \tag{4.24}$$

[7]Actually we can just take any decomposition (4.21) into a difference of positive Radon measures.

Here we need to recall from Proposition 3.2 that continuous functions on X, a locally compact metric space, are (Borel) measurable. Furthermore, we know that there is an increasing sequence of simple functions with limit f, so

$$\left|\int_X f\,d\mu\right| \le \mu(X)\cdot\|f\|_\infty\,. \tag{4.25}$$

This shows that u in (4.24) *is* continuous and that its norm $\|u\| \le \mu(X)$. In fact

$$\|u\| = \mu(X)\,. \tag{4.26}$$

Indeed, the inner regularity of μ implies that there is a compact set $K \Subset X$ with $\mu(K) \ge \mu(X) - \frac{1}{n}$; then there is $f \in \mathcal{C}_c(X)$ with $0 \le f \le 1$ and $f = 1$ on K. It follows that $\mu(f) \ge \mu(K) \ge \mu(X) - \frac{1}{n}$, for any n. This proves (4.26).

We still have to show that if u is defined by (4.24), with μ a finite positive Radon measure, then the measure $\tilde{\mu}$ defined from u via (4.24) is precisely μ itself.

This is easy provided we keep things clear. Starting from $\mu \ge 0$ a finite Radon measure, define u by (4.24) and, for $U \subset X$ open

$$\tilde{\mu}(U) = \sup\left\{\int_X f d\mu,\ 0 \le f \le 1,\ f \in C(X),\ \operatorname{supp}(f) \Subset U\right\}. \tag{4.27}$$

By the properties of the integral, $\tilde{\mu}(U) \le \mu(U)$. Conversely if $K \Subset U$ there exists an element $f \in \mathcal{C}_c(X)$, $0 \le f \le 1$, $f = 1$ on K and $\operatorname{supp}(f) \subset U$. Then we know that

$$\tilde{\mu}(U) \ge \int_X f d\mu \ge \mu(K). \tag{4.28}$$

By the inner regularity of μ, we can choose $K \Subset U$ such that $\mu(K) \ge \mu(U) - \epsilon$, given $\epsilon > 0$. Thus $\tilde{\mu}(U) = \mu(U)$.

This proves the Riesz representation theorem, modulo the decomposition of the measure - which I will do in class if the demand is there! In my view this is quite enough measure theory. $\square$

Notice that we have in fact proved something stronger than the statement of the theorem. Namely we have shown that under the correspondence $u \longleftrightarrow \mu$,

$$\|u\| = |\mu|\,(X) =: \|\mu\|_1\,. \tag{4.29}$$

Thus the map is an *isometry*.