MATH 18.152 - PROBLEM SET 7

18.152 Introduction to PDEs, Fall 2011 .... Professor: Jared Speck

Problem Set 7, **Due: at the start of class on 11-3-11**

I. Let $R > 0$, and let $f(x)$, $g(x)$ be smooth functions on $\mathbb{R}^3$ that vanish outside of $B_R(0) \overset{\text{def}}{=}$ the ball of radius R centered at the origin. Let $u(t, x)$ be the corresponding unique solution to the following global Cauchy problem on $\mathbb{R}^{1+3}$:

$$
\begin{aligned}
(0.0.1) \qquad\qquad -\partial_t^2 u(t, x) + \Delta u(t, x) &= 0, \\
u(0, x) &= f(x), \\
\partial_t u(0, x) &= g(x).
\end{aligned}
$$

Show that there exists a constant $C > 0$ depending on R, $\sup_{y \in B_R(0)} |f(y)|$, $\sup_{y \in B_R(0)} |\nabla f(y)|$, and $\sup_{y \in B_R(0)} |g(y)|$, (but not on t or x) such that $|u(t, x)| \leq \frac{C}{1+t}$ holds for all $t \geq 0$ and all $x \in \mathbb{R}^3$. Do not try to come up with a very precise estimate for C; be as lazy as you can while still being correct!

Hint: Estimate the integrals that appear in Kirchhoff's formula. In particular, this will require that you estimate how large the surface area of the intersection of $\partial B_t(x)$ with $B_R(0)$ can be. Psychologically, it may be easier to divide the estimates into two cases: $t \leq 2R$ and $t \geq 2R$.

Remark 0.0.1. This estimate is called a "decay estimate" for the linear wave equation, and the rate $\frac{C}{1+t}$ is sharp in 3 spatial dimensions. In n spatial dimension, it can be shown that the sharp decay rate is $\frac{C}{(1+t)^{(n-1)/2}}$ (there is no decay when $n = 1$!). This phenomenon is connected to *dispersion*: the more spatial dimensions there are, the more opportunity that waves have to spread out. This rate and mechanism of decay are very different than the ones we encountered in our study of solutions to the heat equation on $\mathbb{R}^{1+n}$.

II. Assume that the matrices Λ and Ω are Lorentz transformations. Recall that this means that

$$
\begin{aligned}
(0.0.2\text{a}) \qquad\qquad \Lambda^T m \Lambda &= m, \\
(0.0.2\text{b}) \qquad\qquad \Omega^T m \Omega &= m,
\end{aligned}
$$

where $m_{\mu\nu} = \text{diag}(-1, 1, 1, \cdots, 1)$ denotes the Minkowski metric on $\mathbb{R}^{1+n}$, and T denotes the matrix transpose.

a) Using elementary properties of the determinant, show that $|\det\Lambda| = 1$.

b) Show that the matrix product $\Lambda\Omega$ is also a Lorentz transformation.

c) Show that the matrix inverse Λ^{-1} is also a Lorentz transformation.

III. Consider the following "boost linear transformation in the x^1 direction" on $\mathbb{R}^{1+3}$:

$$(0.0.3) \qquad \Lambda^{\alpha}{}_{\mu} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $v \in (-1, 1)$ is a "velocity" and $\gamma \overset{\text{def}}{=} \sqrt{\frac{1}{1-v^2}}$. Show that Λ is a Lorentz transformation.

Remark 0.0.2. Analogous boost transformations exist in any number of spatial dimensions.

IV. Let X be a future-directed time-like vector in $1 + n-$dimensional Minkowski spacetime, where $m_{\mu\nu} = (m^{-1})^{\mu\nu} = \text{diag}(-1, 1, 1, \cdots, 1)$. Recall that this means that $m(X, X) < 0$ and $X^0 > 0$. Show that there exists a proper Lorentz transformation ("proper" means that its determinant is equal to 1, rather than -1) Λ such that $(\Lambda X)^{\mu} = (c_1, 0, 0, 0, \cdots, 0)$ for some number $c_1 > 0$. This shows that if X is future-directed and timelike, you can always pick a standard rectangular Minkowskian coordinate system (in which $m_{\mu\nu} = \text{diag}(-1, 1, 1, \cdots, 1)$) such that X is parallel to the t axis.

   Hint: First briefly argue that there exists a product R of spatial rotations (which are Lorentz transformations) such that $(RX)^{\mu} = (c_1, c_2, 0, 0, \cdots, 0)$. Then construct a Lorentz boost B such that $(BRX)^{\mu} = (d, 0, 0, 0, \cdots, 0)$. The transformation of interest is therefore $\Lambda = BR$.

V. Let X, Y be future-directed (i.e., $X^0 > 0$, $Y^0 > 0$) timelike vectors in $1 + n-$dimensional Minkowski spacetime. Recall that by definition, null vectors are vectors L such that $m(L, L) = 0$. Show that there exists a pair of *future-directed* null vectors $L, \underline{L}$ normalized by $m(L, \underline{L}) = -2$ and *positive* constants a, b, c, d such that $X = aL + b\underline{L}$, $Y = cL + d\underline{L}$.

   Hint: By Problem **IV**, you can assume that $X^{\mu} = (X^0, 0, 0, \cdots, 0)$. By performing a further sequence of spatial rotations R, argue that you can also assume that $Y^{\mu} = (Y^0, Y^1, 0, 0, \cdots, 0)$. You can therefore reduce the problem to case of $1 + 1$ spacetime dimensions, where it is relatively easy (quite easy, in fact!) to explicitly find an L and an $\underline{L}$.

18.152 Introduction to Partial Differential Equations.
Fall 2011