Lecture 6

Review

U open $\mathbb{C}^n$. Make the convention that $\Omega^r(U) = \Omega^r$. We showed that $\Omega^r = \bigoplus_{p+q=r} \Omega^{p,q}$, i.e. its bigraded. And we also saw that $d = \partial + \overline{\partial}$, so the coboundary operator breaks up into bigraded pieces.

$$\partial : \Omega^{p,q} \to \Omega^{p+1,q} \qquad \overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$$

$\omega \in \Omega^r, \mu \in \Omega^s$. Then

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^r \omega \wedge d\mu$$

there are analogous formulas for $\partial, \overline{\partial}$

$$\overline{\partial}(\omega \wedge \mu) = \overline{\partial}\omega \wedge \mu + (-1)^r \omega \wedge \overline{\partial}\mu$$

Because of bi-grading the de Rham complex breaks into subcomplexes

$$(1)_q : \Omega^{0,q} \xrightarrow{\partial} \Omega^{1,q} \xrightarrow{\partial} \Omega^{2,q} \xrightarrow{\partial} \cdots$$

$$(2)_p : \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{0,2} \xrightarrow{\overline{\partial}} \cdots$$

The Dolbeault complex is $(2)_0 : \Omega^{0,0} \xrightarrow{\overline{\partial}} \Omega^{0,1}$.

Last week we showed that if U is a polydisk then the Dolbeault complex is acyclic.

Theorem. *If U is a polydisk then complex $(1)_q$ and $(2)_p$ are exact for all p, q.*

Proof. Take $I = (i_1, \dots, i_p)$, define $\Omega_I^{p,q} := \Omega^{0,q} \wedge dz_I$. And $\omega \in \Omega_I^{p,q}$ if and only if $\omega = \mu \wedge dz_I$, $\mu \in \Omega^{0,q}$. And

$$\overline{\partial}(\omega) = \overline{\partial}(\mu \wedge dz_I) = \overline{\partial}\mu \wedge dz_I$$

Therefore, if $\omega \in \Omega_I^{p,q}$, then $\overline{\partial}\omega \in \Omega_I^{p,q+1}$. We can get another complex, define $(2)p_I : \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega_I^{p,1} \xrightarrow{\overline{\partial}} \dots$. Now the map $\mu \in \Omega^{0,q} \mapsto \mu \wedge dz_I$. This maps $(2)_0$ bijectively onto $(2)_I$. So (2) is acyclic. And $\Omega^{p,q} = \bigoplus_I \Omega_I^{p,q}$ implies that $(2)_p$ is acyclic.

What about complex with ∂?

Take $\omega \in \Omega^{p,q}$, then

$$\omega = \sum f_{I,J} dz_I \wedge d\bar{z}_J \qquad f_{I,J} \in C^\infty(U), \quad |I| = p, |J| = q$$

Take complex conjugates

$$\bar{\omega} = \sum \bar{f}_{I,J} d\bar{z}_I \wedge dz_J \in \Omega^{q,p} \qquad \overline{\partial \omega} = \bar{\partial}\bar{\omega}$$

This map $\omega \mapsto \bar{\omega}$ maps $(1)_p$ to $(2)_p$ so $(2)_p$ acyclic implies that $(1)_p$ is acyclic. ☐

The Subcomplex $(A,\ d)$

Another complex to consider. We look at the map $\Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1}$. Denote by A^p the kernel of this map, $\ker\{\Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1}\}$. Suppose $\mu \in A^p$, $\partial\mu \in \Omega^{p+1,0}$, and we know that $\overline{\partial}\partial\mu = -\partial\overline{\partial\mu} = 0$, so $\partial\mu \in A^{p+1}$. Moreover, $d\mu = \partial\mu + \overline{\partial}\mu = \partial\mu$, so we have a subcomplex (A, d) of (Ω, d), the de Rham complex

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots$$

This complex has a fairly simple description. Suppose $\mu \in \Omega^{p,0}$, $\mu = \sum_{|I|=p} f_I dz_I$, and suppose further that $\overline{\partial}\mu = 0$, i.e. $\mu \in A^p$. Then

$$\overline{\partial}\mu = \sum \frac{\partial f_I}{\partial \bar{z}_i} d\bar{z}_i \wedge dz_I = 0 \qquad\qquad \frac{\partial f_I}{\partial \bar{z}_i} = 0 \qquad i = 1, \dots, n$$

so the f_i are holomorphic. Because of this we have the following definition

Definition. The complex (A^*, d) is called the **Holomorphic de Rham complex**.

When is this complex acyclic? To answer this, we go back to the real de Rham complex.

Reminder of Real de Rham Complex

Consider the usual (real) de Rham complex. Let U be an open set in $\mathbb{R}^n$. Then we know

Theorem (Poincare Lemma). *If U is convex then $(\Omega^*(U), d)$ is exact.*

Proof. U convex, and to make things simpler, let $0 \in U$. Let $\rho : U \to U$, $\rho \equiv 0$. Construct a homotopy operator $Q : \Omega^k(U) \to \Omega^{k-1}(U)$, satisfying

$$dQ\omega + Qd\omega = \omega - \rho^*\omega$$

for all $\omega \in \Omega^*(U)$. The exactness follows trivially if we have this operator. Now, what is the operator? We define it the following way.

If $\omega = \sum f_I(x) dx_I$, $f_I \in C^\infty(U)$. Then

$$Q\omega = \sum_{r,I} (-1)^r x_{i_r} \left(\int_0^1 t^{k-1} f_I(tx) dt \right) dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_r}} \wedge \cdots \wedge dx_{i_k}$$

2nd Homework Problem The holomorphic version of this works. Let $U \subseteq \mathbb{R}^{2n} \subseteq \mathbb{C}^n$, convex with $0 \in U$. Take $\omega = \sum_{|I|=k} f_I dz_I$, $f_I \in \mathcal{O}(U)$. Let Q be the same operator (but holomorphic version)

$$Q\omega = \sum_{r,I} (-1)^r z_{i_r} \left(\int_0^1 t^{k-1} f_I(tz) dt \right) dz_{i_1} \wedge \cdots \wedge \widehat{dz_{i_r}} \wedge \cdots \wedge dz_{i_k}$$

Show $Q : A^k \to A^{k-1}$ and $(dQ + Qd)\omega = \omega - \rho^*\omega$. Homework is to check that this all works. □

Theorem. *U a polydisk. Then if $\omega \in \Omega^{1,1}(U)$ and is closed then there exists a C^∞ function f so that $\omega = \partial\overline{\partial} f$. ($f$ is called the potential function of ω).*

This is an important lemma in Kaehler geometry, which we will use later.

Proof. Just diagram chasing:

$$\begin{array}{ccccccccc}
\overline{A}^1 & \xrightarrow{i} & \Omega^{0,1} & \xrightarrow{\partial} & \Omega^{1,1} & \xrightarrow{\partial} & \Omega^{2,1} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \overline{\partial}} & & \uparrow{\scriptstyle \overline{\partial}} & & \uparrow{\scriptstyle \overline{\partial}} & & \\
\overline{A}^0 & \xrightarrow{i} & \Omega^{0,0} & \xrightarrow{\partial} & \Omega^{1,0} & \xrightarrow{\partial} & \Omega^{2,0} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \\
\mathbb{C} & \xrightarrow{i} & A^0 & \xrightarrow{d} & A^1 & \xrightarrow{d} & A^2 & \longrightarrow & \cdots
\end{array}$$

let $\omega = \omega^{1,1} \in \Omega^{1,1}$, $d\omega = 0$, so $\partial\omega = \overline{\partial}\omega = 0$. $\overline{\partial}\omega = 0$ implies there is an a so that $\omega = \overline{\partial} a, a \in \Omega^{1,0}$. We can find $b \in A^1$ so that $\partial a = \partial b$. So $\partial(a-b) = 0$, and $a - b = \partial c$, where $c \in \Omega^{0,0} = C^\infty$. Then $\overline{\partial}(a-b) = \overline{\partial}\partial c$. Put $\overline{\partial}(a-b) = \overline{\partial a} = \omega$. So $\omega = \overline{\partial}\partial c$. □

Exercise (not to be handed in) $\omega \in \Omega^{p,q}(U)$. And $d\omega = 0$ then $\omega = \overline{\partial}\partial u$, $u \in \Omega^{p-1,q-1}$.

Functoriality

U open in $\mathbb{C}^n$, V open in $\mathbb{C}^k$. Coordinatized by $(z_1, \ldots, z_n)$, $(w_1, \ldots, w_k)$. Let $f : U \to V$ be a mapping, $f = (f_1, \ldots, f_k)$, $f_i : U \to \mathbb{C}$. f is holomorphic if each f_i is holomorphic.

Theorem. *f is holomorphic iff $f^*(\Omega^{1,0}(V) \subseteq \Omega^{1,0}(U)$, i.e. for every $\omega \in \Omega^{1,0}(V)$, $f^*\omega \in \Omega^{1,0}(U)$.*

Proof. Necessity. $\omega = d\omega_i$, then

$$f^*\omega = df_i = \partial f_i + \overline{\partial} f_i \in \Omega^{1,0}(U)$$

then $\overline{\partial} f_i = 0$, so $f_i \in \mathcal{O}(U)$.

Sufficiency. Check this. □

Corollary. *f holomorphic. Then $f^*\Omega^{p,q}(V) \subseteq \Omega^{p,q}(U)$, also $\omega \in \Omega^{p,q}(V)$, then $f^*d\omega = df^*\omega$, which implies that $f^*\partial\omega = \partial f^*\omega$, $f^*\overline{\partial}\omega = \overline{\partial} f^*\omega$.*