18.112 Functions of a Complex Variable
Fall 2008

Solution for 18.112 ps 2

1(Prob 7 on P58).
Solution: Let A be a discrete set in separable metric space (X, d), then for any $a \in A$, there is $r_a > 0$ such that

$$A \cap N_{r_a}(a) = \{a\}.$$

It follows that for any $a \neq b$ in A,

$$N_{r_a/2}(a) \cap N_{r_b/2}(b) = \varnothing.$$

Let E be a countable dense subset, so we can find

$$e_a \in E \cap N_{r_a/2}(a)$$

for each $a \in A$, and

$$e_a \neq e_b$$

for $a \neq b$. So we get an injection from A to E, and thus A is countable.

(Another way.) Subset of a separable metric space is still separable metric space. (It is not obvious. Prove it!) So A is separable. But the only dense subset of A is A itself, since A is discrete. So A is countable.

2(Prob 1 on P66).
Solution: Two simple examples:

$$f_1(re^{i\theta}) = \frac{re^{i\theta}}{1-r}$$
$$f_2(re^{i\theta}) = \tan(\pi r/2)e^{i\theta}.$$

It's not hard to check that they are one-to-one and onto and continuous, and their inverses are continuous.

More examples: take any topological map g maps $[0,1)$ to $[0,\infty)$, and any topological map h maps

$$S^1 = \{e^{i\theta} | \theta \in [0, 2\pi)\}$$

to itself, then

$$f(re^{i\theta}) = g(r)h(e^{i\theta})$$

is a topological map from D to $\mathbb{C}$. For example,

$$f(re^{i\theta}) = \frac{7r^8 e^{i(\theta+\pi)}}{1-r^5}.$$

3(Prob 3 on P66).

Solution: Let $f : X \to Y$ be continuous and one-to-one, where X is compact. Then $f(X)$ is compact, and f^{-1} is well-defined on $f(X)$. We only need to prove that f^{-1} is continuous, i.e. $f = (f^{-1})^{-1}$ maps closed set to closed set. This is true, since for any closed subset $A \in X$, A is compact(since X is compact), thus $f(A)$ is compact, thus $f(A)$ is closed.

4(Prob 4 on P66).

Solution: Let

$$d_0 = \inf\{d(x,y) | x \in X, y \in Y\}.$$

For any $n \in \mathbb{N}$,take $x_n \in X, y_n \in Y$ such that

$$d(x_n, y_n) < d_0 + 1/n.$$

Since X is compact, there is a subsequence $\{x_{n_i}\}$ which converges to a point $x_0 \in X$. Since Y is also compact, there is a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges to a point $y_0 \in Y$. Now

$$\begin{aligned} d_0 &\le d(x_0, y_0) \\ &\le d(x_0, x_{n_{i_j}}) + d(y_0, y_{n_{i_j}}) + d(x_{n_{i_j}}, y_{n_{i_j}}) \\ &\le d_0 + 3/n \end{aligned}$$

for j (and thus n_{i_j}) big enough. So

$$d(x_0, y_0) = d_0.$$

(Another way.) Prove that d is a continuous function on the product space $X \times Y$ by triangular inequality, and prove the product space $X \times Y$ is compact metric space by constructing a metric

$$\tilde{d}((x_1, y_1), (x_2, y_2)) = d(x_1, y_1) + d(x_2, y_2)$$

and then use Theorem 7 in page 62 to prove compactness.

5(Prob 3 on P72).

Solution: It is easy to see that

$$|f(z)^2 - 1| < 1$$

implies

$$\text{Re} f(z) \neq 0.$$

Since $\text{Re} f(z)$ is continuous, thus maps connect set Ω to connect set in $\mathbb{R}$ which does not contain 0. So

$$\text{Re} f(z) > 0 \text{ or } \text{Re} f(z) < 0$$

throughout Ω.