Lecture 11

We review some basic properties of the Riemann integral.

Let $Q \subseteq \mathbb{R}^n$ be a rectangle, and let $f, g : q \to \mathbb{R}$ be bounded functions. Assume that f, g are R. integrable. We have the following properties of R. integrals:

- Linearity: $a, b \in \mathbb{R} \implies af + bg$ is R. integrable and

$$\int_Q af + bg = a \int_Q f + b \int_Q g. \tag{3.82}$$

- Monotonicity: If $f \leq g$, then

$$\int_Q f \leq \int_Q g. \tag{3.83}$$

- Maximality Property: Let $h : Q \to \mathbb{R}$ be a function defined by $h(x) = \max(f(x), g(x))$.

Theorem 3.14. *The function h is R. integrable and*

$$\int_Q h \geq \max\left(\int_Q f, \int_Q g\right). \tag{3.84}$$

Proof. We need the following lemma.

Lemma 3.15. *If f and g are continuous at some point $x_0 \in Q$, then h is continuous at x_0.*

Proof. We consider the case $f(x_0) = g(x_0) = h(x_0) = r$. The functions f and g are continuous at x_0 if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ and $|g(x) - g(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

Substitute in $f(x_0) = g(x_0) = r$. The value of $h(x)$ is either $f(x)$ or $g(x)$, so $|h(x) - r| < \epsilon$ for $|x - x_0| < \delta$. That is $|h(x) - h(x_0)| < \epsilon$ for $|x - x_0| < \delta$, so h is continuous at x_0.

The proofs of the other cases are left to the student.

□

We defined $h = \max(f, g)$. The lemma tells is that h is integrable.

Define E, F, and G as follows:

$$E = \text{ Set of points in } Q \text{ where } f \text{ is discontinuous,} \tag{3.85}$$
$$F = \text{ Set of points in } Q \text{ where } g \text{ is discontinuous,} \tag{3.86}$$
$$G = \text{ Set of points in } Q \text{ where } h \text{ is discontinuous.} \tag{3.87}$$

The functions f, g are integrable over Q if and only if E, F are of measure zero. The lemma shows that $G \subseteq E \cup F$, so h is integrable over Q. To finish the proof, we notice that

$$h = \max(f, g) \geq f, g. \tag{3.88}$$

Then, by monotonicity,

$$\int_Q h \geq \max\left(\int_Q f, \int_Q g\right). \tag{3.89}$$

□

Remark. Let $k = \min(f, g)$. Then $k = -\max(-f, -g)$. So, the maximality property also implies that k is integrable and

$$\int_Q k \leq \min\left(\int_Q f, \int_Q g\right). \tag{3.90}$$

A useful trick for when dealing with functions is to change the sign. The preceding remark and the following are examples where such a trick is useful.

Let $f : Q \to \mathbb{R}$ be a R. integrable function. Define

$$f_+ = \max(f, 0), \quad f_- = \max(-f, 0). \tag{3.91}$$

Both of these functions are R. integrable and non-negative: $f_+, f_- \geq 0$. Also note that $f = f_+ - f_-$. This decomposition is a trick we will use over and over again.

Also note that $|f| = f_+ + f_-$, so $|f|$ is R. integrable. By monotonicity,

$$\begin{aligned} \int_Q |f| &= \int_Q f_+ + \int_Q f_- \\ &\geq \int_Q f_+ - \int_Q f_- \\ &= \int_Q f. \end{aligned} \tag{3.92}$$

By replacing f by $-f$, we obtain

$$\begin{aligned} \int_Q |f| &\geq \int_Q -f \\ &= -\int_Q f. \end{aligned} \tag{3.93}$$

Combining these results, we arrive at the following claim

Claim.

$$\int_Q |f| \geq \left|\int_Q f\right| \tag{3.94}$$

Proof. The proof is above. □

3.6 Integration Over More General Regions

So far we've been defining integrals over rectangles. Let us now generalize to other sets.

Let S be a bounded set in $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ be a bounded function. Let $f_S : \mathbb{R}^n \to \mathbb{R}$ be the map defined by

$$f_S(x) = \begin{cases} f(x) & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases} \tag{3.95}$$

Let Q be a rectangle such that $\text{Int } Q \supset \bar{S}$.

Definition 3.16. The map f is *Riemann integrable over S* if f_S is Riemann integrable over Q. Additionally,

$$\int_S f = \int_Q f_S. \tag{3.96}$$

One has to check that this definition does not depend on the choice of Q, but we do not check this here.

Claim. *Let S be a bounded set in $\mathbb{R}^n$, and let $f, g : S \to \mathbb{R}$ be bounded functions. Assume that f, g are R. integrable over S. Then the following properties hold:*

- *Linearity: If $a, b \in \mathbb{R}$, then $af + bg$ is R. integrable over S, and*

$$\int_S af + bg = a \int_S f + b \int_S g. \tag{3.97}$$

- *Monotonicity: If $f \leq g$, then*

$$\int_S f \leq \int_S g. \tag{3.98}$$

- *Maximality: If $h = \max(f, g)$ (over the domain S), then h is R. integrable over S, and*

$$\int_S h \geq \max\left(\int_S f, \int_S g\right). \tag{3.99}$$

Proof. The proofs are easy, and we outline them here.

- Linearity: Note that $af_S + bg_S = (af + bg)_S$, so

$$\begin{aligned} \int_S af + bg &= \int_Q (af + bg)_S \\ &= a \int_Q f_S + b \int_Q g_S \\ &= a \int_S f + b \int_S g. \end{aligned} \tag{3.100}$$

- Monotonicity: Use $f \leq g \implies f_S \leq g_S$.
- Maximality: Use $h = \max(f, g) \implies h_S = \max(f_S, g_S)$.

□

Let's look at some nice set theoretic properties of the Riemann integral.

Claim. *Let S, T be bounded subsets of $\mathbb{R}^n$ with $T \subseteq S$. Let $f : S \to \mathbb{R}$ be bounded and non-negative. Suppose that f is R. integrable over both S and T. Then*

$$\int_T f \leq \int_S f. \tag{3.101}$$

Proof. Clearly, $f_T \leq f_S$. Let Q be a rectangle with $\bar{S} \supseteq \text{Int } Q$. Then

$$\int_Q f_T \leq \int_Q f_S. \tag{3.102}$$

□

Claim. *Let S_1, S_2 be bounded subsets of $\mathbb{R}^n$, and let $f : S_1 \cup S_2 \to \mathbb{R}$ be a bounded function. Suppose that f is R. integrable over both S_1 and S_2. Then f is R. integrable over $S_1 \cap S_2$ and over $S_1 \cup S_2$, and*

$$\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f. \tag{3.103}$$

Proof. Use the following trick. Notice that

$$f_{S_1 \cup S_2} = \max(f_{S_1}, f_{S_2}), \tag{3.104}$$
$$f_{S_1 \cap S_2} = \min(f_{S_1}, f_{S_2}). \tag{3.105}$$

Now, choose Q such that

$$\text{Int } Q \supset \overline{S_1 \cup S_2}, \tag{3.106}$$

so $f_{S_1 \cup S_2}$ and $f_{S_1 \cap S_2}$ are integrable over Q.

Note the identity

$$f_{S_1 \cup S_2} = f_{S_1} + f_{S_2} - f_{S_1 \cap S_2}. \tag{3.107}$$

So,

$$\int_Q f_{S_1 \cup S_2} = \int_Q f_{S_1} + \int_Q f_{S_2} - \int_Q f_{S_1 \cap S_2}, \tag{3.108}$$

from which it follows that

$$\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f. \tag{3.109}$$

□