18.100B : Fall 2010 : Section R2
Homework 1

Due Tuesday, September 14, 11am

Reading: Thu Sept.9 : ordered sets and fields, Rudin 1.1-31 .

1 . Suppose that S is a set and $\preceq$ is a relation on S with the following properties:

- For all $x \in S$, $x \preceq x$.
- For all $x, y \in S$, if $x \preceq y$ and $y \preceq x$ then $x = y$.
- For all $x, y, z \in S$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

Define a new relation $\prec$ on S by $x \prec y$ iff $x \preceq y$ and $x \neq y$. Does this define an order conforming to Definition 1.5 in Rudin? If so, prove it; if not, exhibit a counterexample.

2 . Exercise 6, p. 22 of Rudin. [Here $b > 1$ is an element of $\mathbb{R}$ and you may use the 'definition' of $\mathbb{R}$ as ordered field with least upper bound property. Then recall that $y = x^{\frac{1}{n}}$ is defined as solution of $y^n = x, y \geq 0$.]

3 . (Exercise 9 p. 22 of Rudin – lexicographic order) For complex numbers $z = a + bi \in \mathbb{C}$ and $w = c + di \in \mathbb{C}$ define "$z < w$" if either $a < c$ or if ($a = c$ and $b < d$). Prove that this turns $\mathbb{C}$ into an ordered set. Is this an ordered field? Does it have the least-upper-bound property?

4 . (a) Prove that the field $\mathbb{Q}$ of rational numbers has the Archimedean property.

(b) Compare the least upper bound property with the Archimedean property – which one is 'stronger'? Why?

5 . Review the logic of a proof by induction. (The – not always reliable – wikipedia gives a good explanation in this case.)

(a) Prove that $(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$ for each $n \in \mathbb{N}$.

(b) Find a proof of the Bernoulli inequality:

$$(1 + x)^n \geq 1 + nx \qquad \text{for all } x \in \mathbb{R}, x \geq -1 \text{ and } n \in \mathbb{N}, n \geq 2.$$

(not for credit) Show that strict inequality $(1 + x)^n > 1 + nx$ holds unless $x = 0$.