**18.100A Practice Problems for Exam 2 F-12** 120 minutes (exam is 80 min.)

**Directions:** You can only use the book; cite relevant theorems when asked to.

**1.** A function $f(x)$ has three distinct zeros $a_0 < a_1 < a_2$ on an interval $I$, and in addition $f'(a_2) = 0$. Assume $f(x)$ has a third derivative $f'''(x)$ at all points of $I$.

Prove there is a point $c \in I$ such that $f'''(c) = 0$.

**2.** Let $f(x) = \dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$. Find the best lower and upper quadratic estimates of the form $1 - x/2 + Kx^2$, valid on $[0,3]$, given by the Linearization Error Theorem.

**3.** For what values of the constant $k > 0$ will the equation $\dfrac{\ln x}{x} = k$ have no solutions?

(You can use instead the alternative form $\ln x = kx$; in either case, make a good drawing, show calculations and brief reasoning; cite relevant theorems.

**4.** Prove from a definition of continuity: $f(x) = \displaystyle\int_0^1 \frac{dt}{1+xt^4}$ is continuous at 0.

**5.** a) A function $f(x)$ is defined on all of $\mathbf{R}$, and its secants (i.e., line segments joining two arbitrary points $(x', f(x'))$ and $(x'', f(x''))$ on its graph) have bounded slope.

Prove $f(x)$ is uniformly continuous on $I$.

(Note that $I$ is not compact, and $f(x)$ is not assumed to be differentiable.)

b) Give an example of an $f(x)$ which satisfies all the hypotheses of part (a) but is not differentiable on $I$; give with proof explicit bounds on the secant slopes of $f(x)$.

**6.** Using the two different methods given below, prove that if $f(t)$ is continuous on $[a,b]$,

$$\int_a^b f(t)\,dt \;=\; f(c)(b-a) \quad \text{for some } c \text{ between } a \text{ and } b.$$

Cite theorems used for each step, and indicate where the continuity of $f(t)$ is being used.

a) Prove it by considering the function $F(x) = \displaystyle\int_a^x f(t)\,dt$.

b) Prove it by using the intermediate value theorem, making use of points $t'$ and $t''$ where $f(t)$ attains respectively its minimum and maximum values on $[a,b]$.

(Assume say $t' < t''$; begin by writing the resulting inequalities $f(t)$ satisfies on $[a,b]$.)

**7.** Let $f(x)$ be continuous on $[0,\infty)$, and assume $\displaystyle\lim_{x\to\infty} f(x) = L \quad (L \neq \pm\infty)$.

Prove $f(x)$ is bounded on $[0,\infty)$. (Cite theorems; note that the interval is not compact.)

**8.** Let $f(x)$ be an integrable function on the interval $[a,b]$ of positive length, and assume that $f(x) = 0$ whenever $x$ is a rational number. Prove that $\displaystyle\int_a^b f(x)\,dx = 0$.

(Nothing is known about the value of $f(x)$ when $x$ is irrational. For a weaker result, you can add the assumption: $f(x) \geq 0$ for all $x \in [a,b]$.)

**9.** For what values of $k$ does $\displaystyle\int_{0+}^{1-} \frac{x^k}{\sqrt{x - x^3}}\,dx$ converge? Give reasoning, cite theorems.

**10.** Suppose $f'(x)$ exists on $[0,\infty)$ and is continuous, $f(0) = 0$, and $0 \leq f(x) \leq e^{kx}$ for some constant $k < 1$. Citing theorems, prove $\displaystyle\int_0^\infty f'(x)\,e^{-x}dx = \int_0^\infty f(x)\,e^{-x}dx$ .