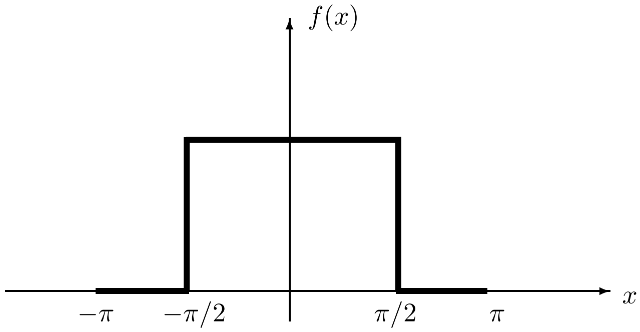
18.085 Computational Science and Engineering I
Fall 2008

18.085 - Mathematical Methods for Engineers I Prof. Gilbert Strang

Solutions - Problem Set 8

Section 4.1

3) $f(x) = \begin{cases} 1 & , \ |x| < \dfrac{\pi}{2} \\ 0 & , \ \dfrac{\pi}{2} < |x| < \pi \end{cases}$



$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\cdot dx \\ &= \frac{1}{2\pi}[\pi/2+\pi/2] \\ &= \frac{1}{2}\,_{\#} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin kx\,dx \\ &= \frac{1}{\pi}\left[-\frac{\cos kx}{k}\right]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{\pi k}\left[-\cos\frac{k\pi}{2}+\cos\left(-\frac{k\pi}{2}\right)\right] \\ &= 0\,_{\#} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos kx\,dx \\ &= \frac{1}{\pi}\left[\frac{\sin kx}{k}\right]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{\pi k}\left[\sin\frac{k\pi}{2}-\sin k\left(-\frac{\pi}{2}\right)\right] \\ &= \frac{2}{\pi k}\sin\left(\frac{k\pi}{2}\right)_{\#} \end{aligned}$$

10) Boundary condition for Laplace's equation

$$u_0 = \begin{cases} 1 & , \ 0 < \theta < \pi \\ 0 & , \ -\pi < \theta < 0 \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} 1\,dx = \frac{1}{2\pi}(\pi - 0) = \frac{1}{2}$$

$$a_k = \frac{1}{\pi}\int_0^{\pi} \cos kx\,dx = \frac{1}{\pi k}\big[\sin kx\big]_0^{\pi} \quad = \frac{\sin k\pi}{\pi k} = 0\ \forall k$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} \sin kx\,dx = \frac{1}{\pi k}\big[-\cos kx\big]_0^{\pi} = \frac{1}{\pi k}[1 - \cos k\pi]$$

$$= \begin{cases} \dfrac{2}{\pi k} & , \ k \text{ odd} \\ 0 & , \ k \text{ even} \end{cases}$$

$$\therefore\ u_0(\theta) = \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

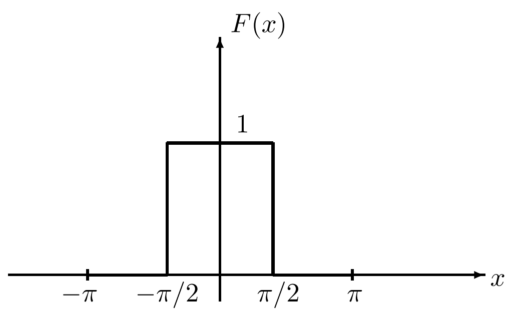
$$\Rightarrow u(r,\theta) = \frac{1}{2} + \frac{2}{\pi}\left[r\sin\theta + \frac{r^3\sin 3x}{3} + \frac{r^5\sin 5x}{5} + \cdots\right]_{\#}$$

At origin, $r = 0,\ \theta = 0$

$$u(0,0) = \frac{1}{2} + \frac{2}{\pi}[0 + 0\ldots]$$
$$= \frac{1}{2}_{\#}$$

13) width $h = \pi$,

a) $$\int |F(x)|^2 dx$$
$$= \int_{-\pi/2}^{\pi/2} 1^2 dx$$
$$= \pi_{\#}$$



b) $$C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,e^{-ikx}\,dx$$
$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-ikx}\,dx$$
$$= \frac{1}{2\pi}\left[\frac{e^{-ikx}}{-ik}\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{2\pi}\left(\frac{1}{ik}\right)\ \big[-e^{-ik\pi/2} + e^{ik\pi/2}\big]$$
$$= \frac{1}{2\pi ki}2i\sin\left(\frac{k\pi}{2}\right)$$

$$= \frac{1}{2\pi}\frac{\sin(k\pi/2)}{(k/2)}$$

$$= \begin{cases} \dfrac{1}{\pi k} & , \ k = \pm 1, 5, 9, \ldots \\ -\dfrac{1}{\pi k} & , \ k = \pm 3, 7, 11, \ldots \\ 0 & , \ k \text{ even} \end{cases}$$

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1dx$$

$$= \frac{1}{2\pi}\left[\pi/2 + \pi/2\right]$$

$$= \frac{1}{2}{}_{\#}$$

c) $$\int_{-\pi}^{\pi} |F(x)|^2\,dx$$

$$= 2\pi\left(|C_0|^2 + |C_1|^2 + |C_{-1}|^2 + \cdots\right)$$

$$= 2\pi\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\pi}\right)^2 + \left(-\frac{1}{\pi}\right)^2 + \left(\frac{1}{3\pi}\right)^2 + \left(-\frac{1}{3\pi}\right)^2 + \cdots\right]$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\underbrace{\left(\frac{\pi^2}{8}\right)}_{\text{Observed from ramp series eqn (15) pg321}}$$

$$= \pi_{\#}$$

If $h = 2\pi$, $\quad C_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} 1\,dx = 1$

$$C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\ dx = \frac{1}{2\pi}\ \frac{\sin(k\pi)}{(k/2)} = 0 \text{ for all } k$$

$\therefore F(x) = 1$ a constant function ${}_{\#}$

18) Heat equation $u_t = u_{xx}$

point source $u(x, 0) = \delta(x)$

with free boundary conditions $u'(\pi, t) = u'(-\pi, t) = 0$

These boundary conditions require cosine series

$$u(x,t) = b_0(t) + \sum_{n=1}^{\infty} b_n(t)\cos nx$$

Substitute into heat equation

$$b_0'(t) + \sum_{n=1}^{\infty} b_n'(t)\cos nx = -\sum_{n=1}^{\infty} b_n(t) n^2 \cos nx$$

since $b_0'(t) = 0, \quad b_n(t) = e^{-n^2 t} b_n(0), \quad b_0(t) = \text{constant}$

Initial condition

$$b_0(0) + \sum_{n=1}^{\infty} b_n(0)\cos\, nx = \delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}[\cos x + \cos 2x + \cos 3x + \cdots]$$

Comparing coefficient

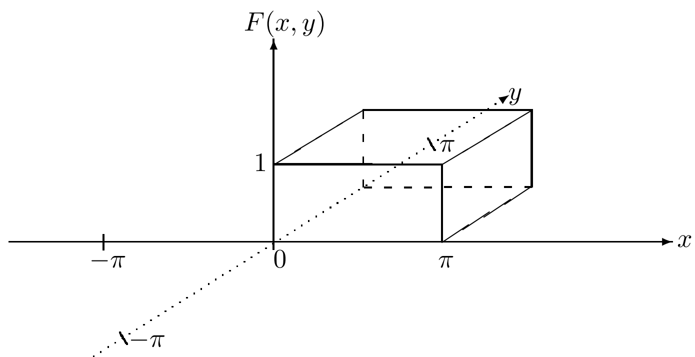
$$b_0(0) = \frac{1}{2\pi}, \quad b_n(0) = \frac{1}{\pi}$$

$$b_0(t) = \frac{1}{2\pi} \quad \text{since } b_0 \text{ is a constant}$$

$$\therefore u(x,t) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi} e^{-n^2 t} \cos\, nx \;\#$$

Section 4.2

a) $F = \text{quarter square} = \begin{cases} 1\,, & 0 \le x \le \pi, \quad 0 \le y \le \pi \\ 0\,, & -\pi < x < 0 \quad \text{or} \quad -\pi < y < 0 \end{cases}$



$$\begin{aligned}
C_{mn} &= \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(x,y) e^{-imx}\, e^{-iny}\, dx\, dy \\
&= \left(\frac{1}{2\pi}\right)^2 \int_{0}^{\pi}\int_{0}^{\pi} e^{-imx}\, e^{-iny}\, dx\, dy \\
&= \left(\frac{1}{2\pi}\right)^2 \int_{0}^{\pi} e^{-iny} \left[\frac{e^{-imx}}{-im}\right]_0^{\pi} dy \\
&= \left(\frac{1}{2\pi}\right)^2 \int_{0}^{\pi} e^{-iny} \left(\frac{1}{im}\right)\left(1 - e^{-im\pi}\right) dy \\
&= \left(\frac{1}{2\pi}\right)^2 \left(\frac{1}{im}\right)\left(1 - e^{-im\pi}\right) \left[\frac{e^{-iny}}{-in}\right]_0^{\pi} \\
&= \left(\frac{1}{2\pi}\right)^2 \left(\frac{1}{im}\right)\left(\frac{1}{in}\right)\left(1 - e^{-im\pi}\right)\left(1 - e^{-in\pi}\right)
\end{aligned}$$

For $m, n \neq 0$

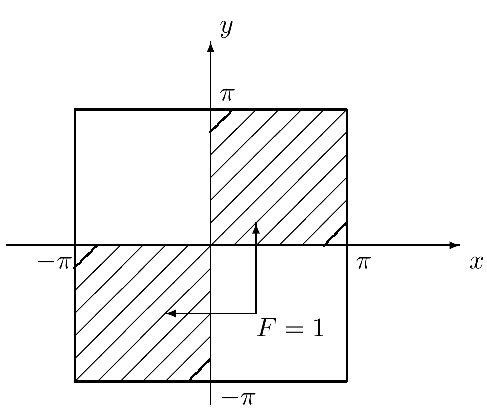
$$C_{mn} = \begin{cases} -\dfrac{1}{\pi^2 mn} & \text{, for } m \text{ and } n \text{ both odd} \\ 0 & \text{, for } m \text{ even or } n \text{ even} \end{cases}$$

$$\begin{aligned} C_\infty &= \left(\frac{1}{2\pi}\right)^2 \int_0^\pi \int_0^\pi 1 \, dx \, dy \\ &= \left(\frac{1}{2\pi}\right)^2 \pi^2 \\ &= \frac{1}{4} \ \# \end{aligned}$$

$$\begin{aligned} C_{0n} &= \left(\frac{1}{2\pi}\right)^2 \int_0^\pi \int_0^\pi e^{-iny} \, dx \, dy \\ &= \left(\frac{1}{2\pi}\right)^2 \int_0^\pi \left[\frac{e^{-iny}}{-in}\right]_0^\pi dx \\ &= \left(\frac{1}{2\pi}\right)^2 \left(\frac{1}{in}\right) \int_0^\pi \left(1 - e^{-in\pi}\right) dx \\ &= \left(\frac{1}{2\pi}\right)^2 \left(\frac{\pi}{in}\right) \left(1 - e^{-in\pi}\right) \\ &= \begin{cases} \dfrac{1}{2\pi i n} & \text{, for } n \text{ odd} \\ 0 & \text{, for } n \text{ even} \ \# \end{cases} \end{aligned}$$

$$\begin{aligned} C_{m0} &= \left(\frac{1}{2\pi}\right)^2 \int_0^\pi \int_0^\pi e^{-imy} \, dx \, dy \\ &= \begin{cases} \dfrac{1}{2\pi i m} & \text{, for } m \text{ odd} \\ 0 & \text{, for } m \text{ even} \ \# \end{cases} \end{aligned}$$

b) $F = \text{ checkerboard } = \begin{cases} 1 & \text{if } xy > 0 \quad -\pi < x \leq \pi \\ 0 & \text{if } xy < 0 \quad -\pi < y \leq \pi \end{cases}$



$$
\begin{aligned}
C_{mn} &= \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(x,y)\; e^{-imx}\; e^{-iny}\; dx\,dy \\
&= \left(\frac{1}{2\pi}\right)^2 \int_{0}^{\pi}\int_{0}^{\pi} e^{-imx}\; e^{-iny}\; dx\,dy + \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{0}\int_{-\pi}^{0} e^{-imx}e^{-iny}\; dx\,dy \\
&= \left(\frac{1}{2\pi}\right)^2 \left\{\int_{0}^{\pi} e^{-iny}\left[\frac{e^{-imx}}{-im}\right]_0^{\pi} dy + \int_{-\pi}^{0} e^{-iny}\left[\frac{e^{-imx}}{-im}\right]_{-\pi}^{0} dy\right\} \\
&= \left(\frac{1}{2\pi}\right)^2 \left\{\left(\frac{1}{im}\right)\left(1-e^{-im\pi}\right)\left[\frac{e^{-iny}}{-in}\right]_0^{\pi} + \left(\frac{1}{im}\right)\left(-1+e^{im\pi}\right)\left[\frac{e^{-iny}}{-in}\right]_{-\pi}^{0}\right\} \\
&= \left(\frac{1}{2\pi}\right)^2 \left(\frac{1}{im}\right)\left(\frac{1}{in}\right)\left\{\left(1-e^{-im\pi}\right)\left(1-e^{-in\pi}\right) + \left(-1+e^{im\pi}\right)\left(-1+e^{in\pi}\right)\right\}
\end{aligned}
$$

For $m, n \neq 0$,

$$
C_m = \begin{cases} -\dfrac{2}{\pi^2 mn}\ , & \text{for } m,\, n \text{ odd} \\[2ex] 0\ , & \text{if } m \text{ even or } n \text{ even} \end{cases}
$$

$$
\begin{aligned}
C_{\infty} &= \left(\frac{1}{2\pi}\right)^2 \left\{\int_{0}^{\pi}\int_{0}^{\pi} dx\,dy + \int_{-\pi}^{0}\int_{-\pi}^{0} dx\,dy\right\} \\
&= \frac{1}{2}\,_{\#}
\end{aligned}
$$

$$
\begin{aligned}
C_{0n} &= \left(\frac{1}{2\pi}\right)^2 \left\{\int_{0}^{\pi}\int_{0}^{\pi} e^{-iny} dx\,dy + \int_{-\pi}^{0}\int_{-\pi}^{0} e^{-iny}\; dx\,dy\right\} \\
&= \left(\frac{1}{2\pi}\right)^2 \left\{\int_{0}^{\pi} \left[\frac{e^{-iny}}{-in}\right]_0^{\pi} dx + \int_{-\pi}^{0} \left[\frac{e^{-iny}}{-in}\right]_{-\pi}^{0} dx\right\} \\
&= \left(\frac{1}{2\pi}\right)^2 \left\{\int_{0}^{\pi} \left(\frac{1}{in}\right)(1-e^{-in\pi})dx + \int_{-\pi}^{0} \left(\frac{1}{in}\right)(-1+e^{in\pi})dx\right\} \\
&= \left(\frac{1}{2\pi}\right)^2 \left(\frac{1}{in}\right)\left\{(1-e^{-in\pi})\pi + (-1+e^{in\pi})\pi\right\} \\
&= \left(\frac{1}{2\pi}\right)^2 \left(\frac{\pi}{in}\right)\left\{(1-e^{-in\pi}) + (-1+e^{in\pi})\right\} \\
&= 0\,_{\#}
\end{aligned}
$$

$$
\begin{aligned}
C_{m0} &= \left(\frac{1}{2\pi}\right)^2 \left\{\int_{0}^{\pi}\int_{0}^{\pi} e^{-imx}\; dx\,dy + \int_{-\pi}^{0}\int_{-\pi}^{0} e^{-imx}\; dx\,dy\right\} \\
&= 0\,_{\#}
\end{aligned}
$$

2) $-S(x,y) = S(-x,-y)$ odd function will have double sine series $\Sigma\Sigma b_{mn} \sin mx \sin ny$

For e.g. $S(x,y) = \begin{cases} \;\;1 & \text{if } xy \geq 0\,, \quad -\pi < x \leq \pi \\ -1 & \text{if } xy < 0\,, \quad -\pi < y \leq \pi \end{cases}$

$\left.\begin{array}{l} S(-x,y) = -S(x,y) \\ S(x,-y) = -S(x,y) \end{array}\right\}$ are also odd functions

Orthogonality

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (\sin kx \sin ly)(\sin mx \sin ny)dx\,dy = 0$$

for $k = m$ and $l = n$

on the square $\quad -\pi \leq x \leq \pi \qquad -\pi \leq y \leq \pi_{\#}$

20) Sturm-Liouville eigenvalue problem

$$(pu')' + qu + \lambda wu = 0$$

$$u_1\left\{(pu_2{}')' + qu_2 + \lambda_1 wu_2\right\} = 0 \quad \text{—— } ①$$

$$u_2\left\{(pu_1{}')' + qu_1 + \lambda_2 wu_1\right\} = 0 \quad \text{—— } ②$$

① − ②

$$u_1(pu_2{}')' - u_2(pu_1{}')' + q\overset{0}{\cancel{(u_1u_2 - u_2u_1)}} + wu_1u_2(\lambda_1 - \lambda_2) = 0 \quad \text{—— } ③$$

$$\therefore u_1(pu_2{}')' - u_2(pu_1{}')' + wu_1u_2(\lambda_1 - \lambda_2) = 0$$

$\displaystyle\int_a^b u_2(pu_1')'dx$ where a, b are boundary

$$= \underbrace{\overset{0}{\cancel{\left[u_2(pu_1')\right]_a^b}}}_{\text{Boundary}} - \int_a^b u_2'(pu_1')dx = -\int_a^b u_2'(pu_1')dx$$

$\displaystyle\int_a^b u_1(pu_2')'dx$

$$= \underbrace{\overset{0}{\cancel{\left[u_1(pu_2')\right]_a^b}}}_{\text{Boundary}} - \int_a^b u_1'(pu_2')dx = -\int_a^b u_1'(pu_2')dx$$

From ③

$$\int_a^b \left\{u_1(pu_2')' - u_2(pu_1')'\right\}dx = \int_a^b (\lambda_2 - \lambda_1)wu_1u_2\; dx$$

$$-\int_a^b u_1'(pu_2')dx + \int_a^b u_2'(pu_1')dx = \int_a^b (\lambda_2 - \lambda_1)wu_1u_2\; dx$$

$$\therefore \int_a^b (\lambda_2 - \lambda_1) w u_1 u_2 \; dx = 0$$

If $(\lambda_2 - \lambda_1) \neq 0$

$$\int_a^b w u_1 u_2 \; dx = 0_{\#} \text{ weighted orthogonality}$$

Section 4.3

2) $\quad f_j = \sum_{k=0}^{N-1} c_k w^{jk} \Rightarrow f = F_N c$

row j^{th} of $\overline{F} = [1 \quad \overline{w}^j \quad \overline{w}^{2j} \; \dots \quad \overline{w}^{j(N-1)}]$

row $(N-j)^{\mathsf{th}}$ of $F = [1 \quad w^{(N-j)} \quad w^{2(N-j)} \; \dots \quad w^{(N-j)(N-1)}]$

$$\begin{aligned} \overline{w}^{j} &= e^{-2\pi i j/N} \\ w^{N-j} &= e^{(2\pi i/N)(N-j)} = e^{2\pi i}\, e^{-2\pi i j/N} \\ &= 1 \cdot e^{-2\pi i j/N} \\ &= \overline{w}^{j} \end{aligned}$$

$$\begin{aligned} \text{Similarly } \overline{w}^{2j} &= w^{2(N-j)} \\ &\;\vdots \\ \overline{w}^{j(N-1)} &= w^{(N-j)(N-1)} \end{aligned}$$

$\therefore$ row j^{th} of $\overline{F}$ is the same as row $(N-j)^{\mathsf{th}}$ of $F_{\#}$

6)

$$F_6 = \left[\begin{array}{ccc:ccc} 1 & & & 1 & & \\ & 1 & & & w & \\ & & 1 & & & w^2 \\ \hdashline 1 & & & -1 & & \\ & 1 & & & -w & \\ & & 1 & & & -w^2 \end{array}\right] \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & & & \\ 1 & w^2 & w^4 & & 0 & \\ 1 & w^4 & w^8 & & & \\ \hdashline & & & 1 & 1 & 1 \\ & 0 & & 1 & w^2 & w^4 \\ & & & 1 & w^4 & w^8 \end{array}\right] \underbrace{\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hdashline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{P}$$

$$= \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w^2 & w^4 & w & w^3 & w^5 \\ 1 & w^4 & w^8 & w^2 & w^6 & w^{10} \\ \hdashline 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & w^2 & w^4 & -w & -w^3 & -w^5 \\ 1 & w^4 & w^8 & -w^2 & -w^6 & -w^{10} \end{array}\right] \left[\begin{array}{cccccc} 1 & & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & 1 & & & & \\ & & & 1 & & \\ & & & & & 1 \end{array}\right]$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -w & w^2 & -w^3 & w^4 & -w^5 \\ 1 & -w^2 & w^4 & -w^6 & w^8 & -w^{10} \end{bmatrix} \qquad \text{Where } D = \begin{bmatrix} 1 & & \\ & w & \\ & & w^2 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w^2 & w^4 \\ 1 & w^4 & w^8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} \\ 1 & w^3 & w^6 & w^9 & w^{12} & w^{15} \\ 1 & w^4 & w^8 & w^{12} & w^{16} & w^{20} \\ 1 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} \end{bmatrix} \text{\# verified}$$

8) $c = (1,0,1,0)$, $\quad N = 4$

$c' = (1,1) \qquad c'' = (0,0)$

$f' = F_2 c' \qquad f'' = F_2 c''$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First half,

$$f_j = f'_j + (w_N)^j f''_j$$

$$f_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & \\ & i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Second half,

$$f_{j+M} = f'_j - (w_N)^j f''_j$$

$$f_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & \\ & i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\therefore \text{Combine } f = [f_1\,;\, f_2] = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \#$$

For $c = (0,1,0,1)$

$c' = (0,0) \qquad c'' = (1,1)$

$f' = F_2 c' \qquad f'' = F_2 c''$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

First half,

$$f_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & \\ & i \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Second half,

$$f_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & \\ & i \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$\therefore$ Combine

$$f = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}_{\#}$$

10) $w = e^{2\pi i/64}$

$w^2 = e^{2\pi i/32} \qquad \sqrt{w} = e^{2\pi i/128}$

$\therefore w^2$ and $\sqrt{w}$ are among the **32** and **128** roots of 1 $_{\#}$

15)
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & i & & \\ & & i^2 & \\ & & & i^3 \end{bmatrix}$$

$\therefore \lambda_1 = 1, \quad \lambda_2 = i, \quad \lambda_3 = i^2, \quad \lambda_4 = i^3{}_{\#}$