18.085 Computational Science and Engineering I
Fall 2008

18.085 - Mathematical Methods for Engineers I Prof. Gilbert Strang

Solutions - Problem Set 1

Section 1.1

2)

$$T_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$U^{\mathrm{T}}U = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = T_3$$

$$UU^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$T_3 = U^{\mathrm{T}}U$$

$$T_3^{-1} = (U^{\mathrm{T}}U)^{-1}$$

$$= (U^{-1})(U^{-1})^{\mathrm{T}}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{\#}$$

5)

$$K_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$K_2^{-1} = \frac{1}{4-1} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}_{\#}$$

Given that

$$K_3^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$K_4^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Since $\mathsf{det}(K_2) = 3$, $\mathsf{det}(K_3) = 4$, $\mathsf{det}(K_4) = 5$

$\therefore$ determinant of $K_5 = 6_{\#}$

From MATLAB,

$\mathsf{det}(K_5) = 6_{\#}$

$$\mathsf{inv}(K_5) = \frac{1}{6}\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}_{\#}$$

$$\mathsf{det}(K) * \mathsf{inv}(K) = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}_{\#}$$

20)

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix}\begin{bmatrix} 2 & 4 \end{bmatrix}$$

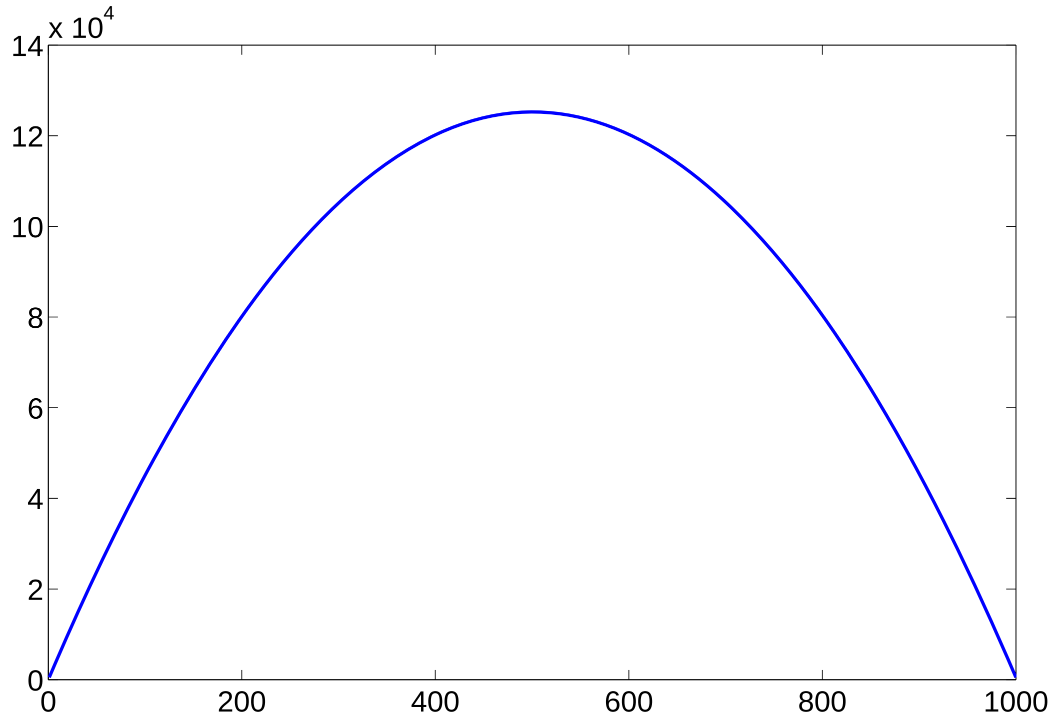
$$= \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 6 & 12 \\ 10 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 16 \\ 14 & 28 \end{bmatrix}_{\#}$$

For columns times row, n by n would require n^3 multiplications

22) MATLAB's code

```
n = 1000;
e = ones(n, 1);
K = spdiags([-e, 2*e, -e], -1:1, n, n);
u = K \ e;
plot(u);
```



Section 1.2

1) $$u(x) = \begin{cases} Ax & \text{if } x \le 0 \\ Bx & \text{if } x \ge 0 \end{cases}$$

$$u''(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} = \delta(x) \ \#$$

$$U_n = \begin{cases} A_n & \text{if } n \le 0 \\ B_n & \text{if } n \ge 0 \end{cases} = \begin{bmatrix} -2A \\ -A \\ 0 \\ B \\ 2B \end{bmatrix}$$

$$\Delta^2 U_n = \begin{bmatrix} \ddots & & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ -3A \\ -2A \\ -A \\ 0 \\ B \\ 2B \\ 3B \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ -A+B \\ 0 \\ 0 \\ \vdots \end{bmatrix}_{\#}$$

2) $-u''(x) = \delta(x)$

$u(-2) = 0,\ u(3) = 0$

$-\int u''(x) = \int \delta(x)$

$-\big[u'(x)\big]_L^R = 1$

$u'_R(x) - u'_L(x) = -1$

Given that $u = \begin{cases} A(x+2) & x \le 0 \\ B(x-3) & x \ge 0 \end{cases}$

$B - A = -1$ —— ①

Since the pieces meet at $x = 0$,

$A(0+2) = B(0-3)$

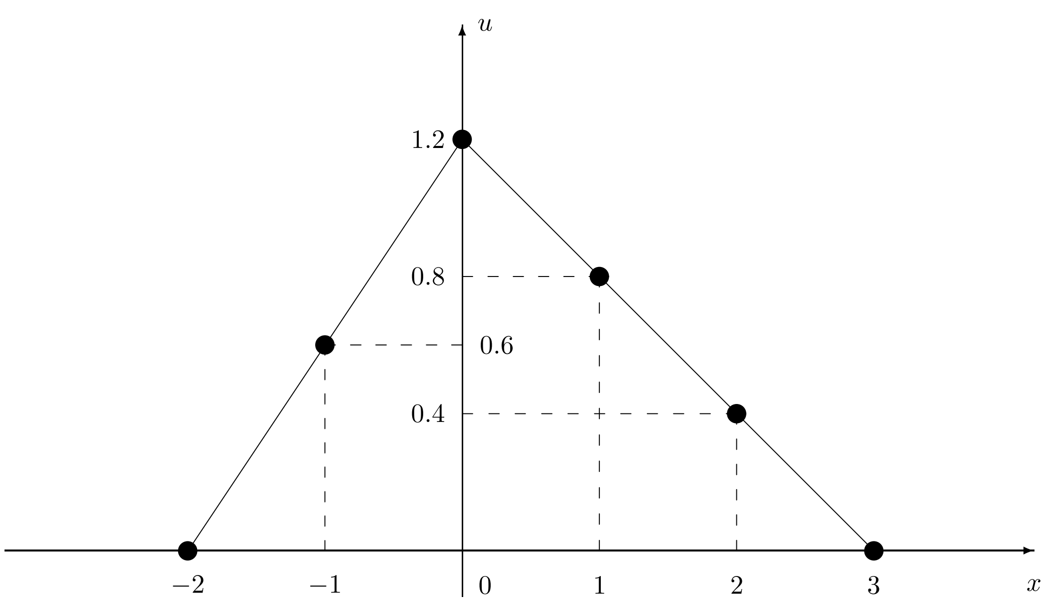
$A = -\frac{3}{2}B$ —— ②

② → ①

$B - \left(-\frac{3}{2}B\right) = -1$

$B = -0.4$
$A = 0.6$

$\therefore u = \begin{cases} 0.6(x+2) & x \le 0 \\ -0.4(x-3) & x \ge 0 \end{cases}$



$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \begin{bmatrix} u(-1) \\ u(0) \\ u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u(-1) \\ u(0) \\ u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \frac{1}{5}$$

$$= \frac{1}{5} \begin{bmatrix} 3 \\ 6 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} u(-1) \\ u(0) \\ u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} 0.6 \\ 1.2 \\ 0.8 \\ 0.4 \end{bmatrix}_{\#}$$

The discrete solution yields the same results as the actual solution $_{\#}$

4) $$\Delta_- = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \\ & & & -1 & 1 \end{bmatrix} \text{Backward difference matrix}$$

$$\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \\ & & & -1 & 1 \end{bmatrix} \overbrace{\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & & & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}}^{\text{sum matrix}}$$

$$= \begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$$= \quad I_{\#}$$

Verified that the inverse of backward difference matrix is the sum matrix $_{\#}$

$$\Delta_0 = \frac{1}{2}\left(\Delta_+ + \Delta_-\right)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix} + \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \\ & & & -1 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & & \ddots & \\ & & & -1 & 0 \end{bmatrix}$$

For $n = 3$

$$\Delta_0 u = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ c \end{bmatrix}$$

For $n = 5$

$$\Delta_0 u = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ c \\ 0 \\ c \end{bmatrix}$$

The nullvector of $\Delta_0 u = 0$ is not the zero vector, therefore Δ_0 is not invertible $_{\#}$

7) $$\frac{-u_2 + 8u_1 - 8u_{-1} + u_{-2}}{12h} = \frac{du}{dx} + bh^4\frac{d^5u}{dx^5} + \cdots$$

For $u(x) = 1$

$$\frac{du}{dx} = \frac{-1 + 8 - 8 + 1}{12h}$$

$$= 0 \ _{\#\,\text{correct}}$$

For $u(x) = x^2$

$$\frac{du}{dx} = \frac{-(x+2h)^2 + 8(x+h)^2 - 8(x-h)^2 + (x-2h)^2}{12h}$$

$$= \frac{(x-2h)^2 - (x+2h)^2 + 8\{(x+h)^2 - (x-h)^2\}}{12h}$$

$$= \frac{(x - \cancel{2h} + x + \cancel{2h})(\cancel{x} - 2h - \cancel{x} - 2h) + 8(x + \cancel{h} + x - \cancel{h})(\cancel{x} + h - \cancel{x} + h)}{12h}$$

$$= \frac{-4h(2x) + 8(2x)(2h)}{12h}$$

$$= 2x \text{ \# correct}$$

For $u(x) = x^4$

$$\frac{du}{dx} = \frac{(x-2h)^4 - (x+2h)^4 + 8\{(x+h)^4 - (x-h)^4\}}{12h}$$

$$= \left\{ \begin{array}{l} [(x-2h)^2 + (x+2h)^2][(x-2h)^2 - (x+2h)^2] \\ \quad + 8\{[(x+h)^2 + (x-h)^2][(x+h)^2 - (x-h)^2]\} \end{array} \right\} / 12h$$

$$= \frac{1}{12h}\left\{ -8hx[(x-2h)^2 + (x+2h)^2] + 8(4xh)[(x+h)^2 + (x-h)^2] \right\}$$

$$= \frac{1}{12h}\left\{ \begin{array}{l} -8hx[x^2 - \cancel{4xh} + 4h^2 + x^2 + \cancel{4xh} + 4h^2] \\ \qquad + 8(4xh)[x^2 + \cancel{2xh} + h^2 + x^2 - \cancel{2xh} + h^2] \end{array} \right\}$$

$$= \frac{1}{12h}\left\{ -8hx[2x^2 + 8h^2] + 8(4xh)[2x^2 + 2h^2] \right\}$$

$$= \frac{-8hx}{12h}\left\{ 2x^2 + \cancel{8h^2} - 8x^2 - \cancel{8h^2} \right\}$$

$$= 4x^3 \text{ \# correct}$$

$$u(x+h) = u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u'''(x) + \frac{1}{24}h^4u''''(x) + \frac{1}{120}h^5u^{(5)}(x) + \ldots$$

$$u(x-h) = u(x) - hu'(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u'''(x) + \frac{1}{24}h^4u''''(x) - \frac{1}{120}h^5u^{(5)}(x) + \ldots$$

$$u(x+2h) = u(x) + 2hu'(x) + \frac{1}{2}(2h)^2u''(x) + \frac{1}{6}(2h)^3u'''(x) + \frac{1}{24}(2h)^4u^{(4)}(x) + \frac{1}{120}(2h)^5u^{(5)}(x) + \ldots$$

$$u(x-2h) = u(x) - 2hu'(x) + \frac{1}{2}(2h)^2u''(x) - \frac{1}{6}(2h)^3u'''(x) + \frac{1}{24}(2h)^4u^{(4)}(x) - \frac{1}{120}(2h)^5u^{(5)}(x) + \ldots$$

$$\frac{-u_2 + 8u_1 - 8_{u-1} + u_{-2}}{12h}$$

$$= \left\{ \begin{array}{l} -4hu'(x) - \dfrac{2}{6}(2h)^3 u'''(x) - \dfrac{2}{120}(2h)^5 u^{(5)}(x) \\ \qquad + 8\left[2hu'(x) + \dfrac{2}{6}h^3 u'''(x) + \dfrac{2}{120}h^5 u^{(5)}(x) + \cdots \right] \end{array} \right\} / 12h$$

$$= \frac{12hu'(x) - \frac{2}{5}h^5 u^{(5)}(x) + \ldots}{12h}$$

$$= u'(x) - \frac{1}{30}h^4 u^{(5)}(x) + \cdots$$

$\therefore$ the coefficient $b = -\dfrac{1}{30}$ #

10) $$\Delta_+ \Delta_- = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \\ & & & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & \ddots & \\ & & & 1 & -1 \end{bmatrix}$$

First row $\longrightarrow$ Boundary row corresponds to $u = 0_{\#}$

Last row $\longrightarrow$ Approximation to $u' = 0_{\#}$

19) $$-\frac{d^2u}{dx^2} + \frac{du}{dx} = 1$$

$$\begin{aligned} u(0) &= 0, \quad u(1) = 0 \\ u(x) &= u_p + u_n \\ &= x + A + Be^x \\ u(0) &= 0 \\ 0 &= 0 + A + Be^0 \end{aligned}$$

$$A = -B \quad \text{——①}$$

$$\begin{aligned} u(1) &= 0 \\ 0 &= 1 + A + Be^1 \quad \text{——②} \end{aligned}$$

① → ②

$$0 = 1 + (-B) + Be^1$$

$$B(1 - e^1) = 1$$

$$B = \frac{1}{1 - e^1} \qquad A = \frac{-1}{1 - e^1}$$

$$\therefore u(x) = x - \frac{1}{1 - e}(1 - e^x)_{\#} \qquad \text{Actual Solution}$$

Discrete centered finite difference solution

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \frac{1}{2h}\begin{bmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & 1 \\ & & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$h = \frac{1}{5}$$

From MATLAB,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.0714 \\ 0.1141 \\ 0.1220 \\ 0.0871 \end{bmatrix}$$

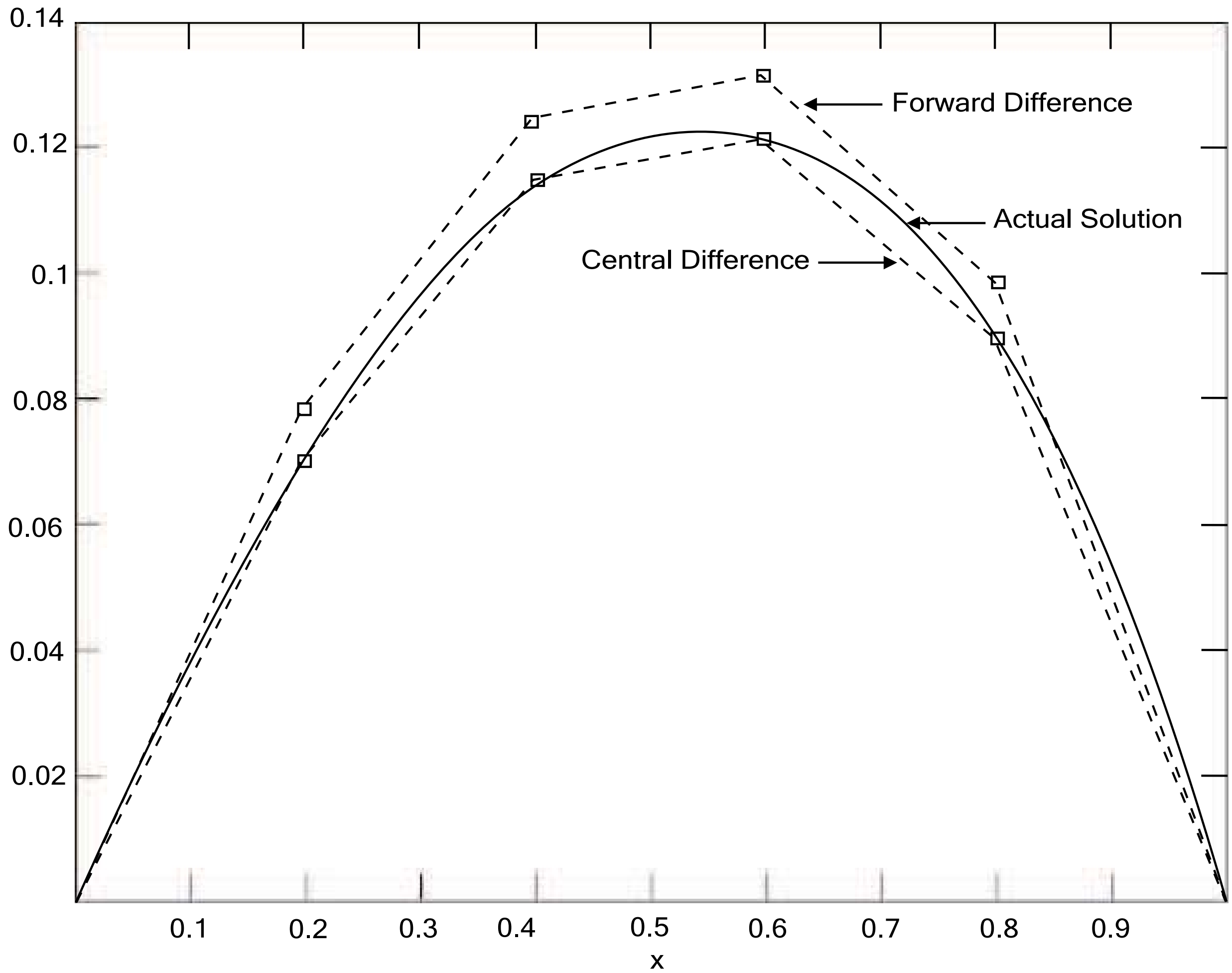
Discrete forward difference for $u'(x)$

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \frac{1}{h}\begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$h = \frac{1}{5}$$

From MATLAB,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.0782 \\ 0.1258 \\ 0.1355 \\ 0.0975 \end{bmatrix}$$



21) $u(h) = u(0) + hu'(0) + \frac{1}{2}h^2u''(0) + \cdots$

$$-u'' = f(x), \quad u'(0) = 0$$

$$\begin{aligned} \frac{u(h) - u(0)}{h} &= \cancelto{0}{u'(0)} + \frac{1}{2}hu''(0) + \cdots \\ \therefore u(1) - u(0) &= \frac{1}{2}h^2u''(0) \ \# \ \text{(Q.E.D)} \end{aligned}$$