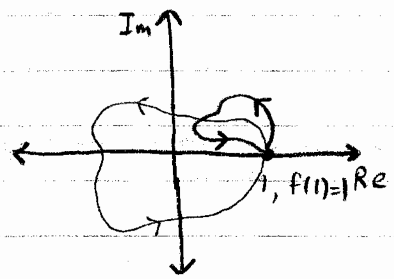
Branch Points and Branch Cuts

ex $f(z) = z^{1/2}$ $\quad f(1) = e^{\frac{1}{2}\operatorname{Ln} z} = e^{\frac{1}{2}[\ln 1 + i(\theta_p + 2\pi k)]} = e^{ik\pi} = \begin{cases} 1, & k=0 \\ -1, & k=1 \end{cases}$



$z = re^{i\theta}$, $f(z) = (re^{i\theta})^{\frac{1}{2}} = \sqrt{r}\, e^{\frac{i\theta}{2}}$ → we eliminated k.
→ continuous change in r and θ
$(\sqrt{r} > 0)$

at $z=1$: $\theta = 0$, $r = 1$, $f(1) = 1$

- you <u>have</u> to determine a starting point

: $\theta = 0 \to \theta = 0$, $f(1) = 1 \to f(1) = 1$

: $\theta = 0 \to \theta = 2\pi$, $f(1) = 1 \to f(1) = -1$

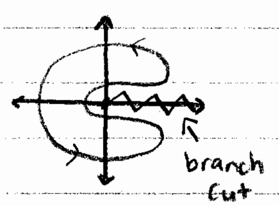
$f(z)$ always gets back the same value if 0 is NOT encircled.
$f(z)$ gets back different values if 0 is encircled.

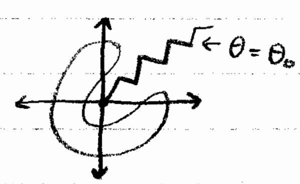
0 is a <u>branch point</u> of $f(z) = z^{1/2}$.

- complex plane consists of 2 "Riemann sheets."

$f(z) = z^{1/2}$: multiple valued function

How can we "make" $f(z)$ single-valued?





$\underbrace{\theta_0 \leq \theta < \theta_0 + 2\pi}_{}$ — 1st Riemann sheet

↓

$f(z)$: single valued

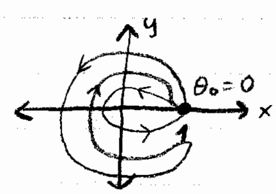
Cross a branch cut → enter 2nd Riemann sheet. $\theta_0 + 2\pi \leq \theta < \theta_0 + 4\pi$

More generally, $f(z) = z^{1/n}$, $n: n \geq 2$ has n Riemann sheets.

ex $f(z) = \operatorname{Ln}(z) = \ln r + i\theta$ ← no restriction on θ

$f(1) = 0 + i\theta_0$ $(0, 2\pi, -2\pi, \ldots)$: infinitely many values

choice: $\theta_0 = 0 \to f(1) = 0$



$z = x + iy$

: $\theta = \theta_0 = 0 \to \theta = 0$; no change in $\operatorname{Ln} z$

: $\theta = \theta_0 = 0 \to \theta = 2\pi$; $\operatorname{Ln}(1) = 0 \to i2\pi$

branch point: $z = 0$. $\operatorname{Ln} z$ has an <u>infinite</u> number of Riemann sheets.

ex inverse sine function: $w = \sin^{-1} z = f(z) \longleftrightarrow z = \sin w$

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} \qquad e^{iw} - e^{-iw} = 2iz$$

$$e^{2iw} - 1 = 2ize^{iw}$$

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0 \leftarrow \text{quadratic}$$

If $R = e^{iw}$, $R^2 - 2izR - 1 = 0$, quadratic equation for R.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(works for complex numbers too)}$$

$$R = iz \underset{\pm?}{+} \sqrt{1 - z^2} = e^{iw}$$

$$\boxed{w = \frac{1}{i} \mathrm{Ln}(iz + \sqrt{1 - z^2})} = \sin^{-1}(z)$$

multiplicity