18.075 Practice Test II for Exam 2

Justify your answers. Cross out what is not meant to be part of your solution.

Total number of points: 60.

I. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 - \pi^2)(x^2 + 1)}\, dx \qquad (1).$$

1. (2pts) By replacing x by the complex variable z, locate and characterize all singularities of the integrand (viewed as a function of z).
2. (2pts) Locate and characterize all the singularities of $\frac{ze^{iz}}{(z^2-\pi^2)(z^2+1)}$, i.e., repeat part (a) after replacing $\sin x = Im(e^{ix})$ by e^{ix}.
3. (6pts) Define the principal value and indented contours in order to evaluate the given integral I of (1) above. In particular, state which parts of the contours finally contribute zero and why.
4. (10pts) Evaluate the requisite integral I by use of the residue theorem.

II. (15pts) Consider the real integral

$$I = \int_{0}^{\infty} \frac{x}{x^4 + 1}\, dx.$$

Evaluate I by use of the residue theorem. **Hint:** Integrate $f(z) = \frac{z}{z^4+1}$ $(z = x + iy)$ around the closed contour consisting of the portions of the x (real) and y (imaginary) axes for which $0 \le x \le R$ and $0 \le y \le R$, and a quadrant of the circle $|z| = R$, and finally let $R \to \infty$.

III. Find the region of convergence of the following series by using the ratio or Cauchy (root) test, where x is real.

1. (4pts) $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$.
2. (4pts) $\sum_{n=0}^{\infty} \frac{n!}{(2n)!}(x+2)^n$.

IV. Locate and classify the singular points of the following differential equations.

1. (2pts) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0$.
2. (3pts) $x^2\frac{d^2y}{dx^2} - (x^2+2)\frac{dy}{dx} - (x+1)y = 0$.

V. Consider the differential equation

$$x^2\frac{d^2y}{dx^2} + (x^2 - x)\frac{dy}{dx} + y = 0.$$

1. (6pts) By substituting $y = \sum_{n=0}^{\infty} A_n x^n$, express the left-hand side of the differential equation as a power series, each term involving the (common) factor x^n.
2. (6pts) Determine the recurrence formula for the coefficients A_n. (You are NOT asked to find the final solution $y(x)$.) How many independent solutions does this method give?