18.075 Solutions to Practice Test 1 for Exam 3

(I.) ① Let $A_n(x) = \dfrac{(x-1)^n}{(n+1)^n}$

Root test: $L = \lim_{n\to\infty} \sqrt[n]{|A_n(x)|} = \lim_{n\to\infty} \dfrac{|x-1|}{n+1} = 0 \cdot |x-1| = 0 < 1$ for every finite x.

Hence, the series converges for all finite x; $R = \infty$.

② Let $A_n(x) = \dfrac{3^n}{2^n+n} x^{3n}$

Root test: $L = \lim_{n\to\infty} \sqrt[n]{|A_n(x)|} = \left(\lim_{n\to\infty} \dfrac{3}{\sqrt[n]{2^n+n}}\right) |x|^3$. Notice that, for $n\to\infty$, $2^n+n \simeq 2^n$.

Hence,

$$L = \frac{3}{2} |x|^3$$

$$\left.\begin{array}{llll} \text{If } \underline{L<1} & \Leftrightarrow & |x| < \left(\frac{2}{3}\right)^{1/3} & \text{series converges} \\ \text{If } \underline{L>1} & \Leftrightarrow & |x| > (2/3)^{1/3} & \text{series diverges} \end{array}\right\} \quad R = \left(\frac{2}{3}\right)^{1/3}.$$

(II.) ① The ODE is written as
$$y'' + \underbrace{\frac{\sin x}{1-\cos x}}_{a_1(x)} y' + \underbrace{\frac{1}{1-\cos x}}_{a_2(x)} y = 0.$$

Possible singularities: $1-\cos x = 0 \Leftrightarrow x = 2n\pi = x_n$ $(n = 0, \pm1, \pm2, \dots)$

Let $t = x - x_n$: $\sin x = \sin t = t - t^3/3! + \dots$ (as $t\to 0$).

$$1-\cos x = 1-\cos t = 1 - (1 - t^2/2! + t^4/4! - \dots) = t^2/2! - t^4/4! + \dots$$

$$a_1(x) = \frac{\sin x}{1-\cos x} = \frac{\sin t}{1-\cos t} = \frac{1 - t^2/3! + \dots}{t(1/2! - t^2/4! + \dots)}$$: has a pole at $t=0$; i.e., at $x = x_n$.

$$a_2(x) = \frac{1}{1-\cos x} = \frac{1}{1-\cos t} = \frac{1}{t^2(1/2! - t^2/4! + \dots)}$$: has a pole at $t=0$.

Hence, $x = x_n = 2n\pi$ are singular points of the ode.

② $(x-\overset{x_0=0}{x_0})\, a_1(x) = \dfrac{x\sin x}{1-\cos x} = \dfrac{1 - x^2/3! + \dots}{1/2! - x^2/4! + \dots}$: has a Taylor series at $x_0=0 \to$ analytic at $x_0=0$.

$(x-\overset{x_0=0}{x_0})^2\, a_2(x) = \dfrac{x^2}{1-\cos x} = \dfrac{1}{1/2! - x^2/4! + \dots}$: has a Taylor series at $x_0=0 \to$ analytic at $x_0=0$.

Hence, $x_0 = 0$ is a regular singular point of this ode.

(III) (1) $y'' - (\ln x)y' + y = 0$

$a_1(x) = -\ln x, \quad a_2(x) = 0.$

Since $x_0 = 0$ is a branch point for $a_1(z)$, $x_0 = 0$ is a singular point of this ODE. This point is an irregular singular point because $z a_1(z)$ is NOT analytic at $x_0 = 0$

(2) $y'' + \dfrac{\sqrt{x}}{\sin\sqrt{x}}\, y' - \dfrac{1}{\sin\sqrt{x}}\, y = 0$

$a_1(x) = \dfrac{\sqrt{x}}{\sin\sqrt{x}}, \quad a_2(x) = \dfrac{1}{\sin\sqrt{x}}$

For $x \to 0$,

$$\sin\sqrt{x} = (\sqrt{x}) - \frac{(\sqrt{x})^3}{3!} + \dots = \sqrt{x}\left(1 - \frac{x}{3!} + \dots\right)$$

$a_1(x) = \dfrac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\,(1 - x/3! + \dots)}$: analytic at $x = 0$

$a_2(x)$: $\dfrac{1}{\sqrt{x}\,(1 - x/3! + \dots)}$: has a branch point at $x_0 = 0$.

Hence, $x_0 = 0$ is a singular point. It is an irregular singular point because

$x^2 a_2(x) = \dfrac{x\sqrt{x}}{1 - x/3! + \dots}$: still has a branch point at $x_0 = 0$ (bec. of $\sqrt{x}$)

(IV) (1) $y'' + \frac{1}{x}(-3)\, y' + \frac{1}{x^2}(3 - x^2)y = 0$

$R(x) = 1, \quad P(x) = -3, \quad Q(x) = 3 - x^2. \quad ; R_0 = 1, P_0 = -3, \quad Q_0 = 3, \quad Q_2 = -1$ (rest 0)

(2) $P_0 = -3, \quad Q_0 = 3$

Indicial equation: $f(s) = s(s-1) + P_0 s + Q_0 = 0 \iff s(s-1) - 3s + 3 = 0$

$\iff s(s-1) - 3(s-1) = 0 \iff (s-1)(s-3) = 0$

$\iff \boxed{s_1 = 3, \quad s_2 = 1}$

(3) $g_n(s) = R_n(s-n)(s-n-1) + P_n(s-n) + Q_n \quad , n \geq 1.$

It follows that $g_n(s) \equiv 0$ except when $n = 2$.

$g_2(s) = R_2(s-2)(s-3) + P_2(s-2) + Q_2 = -1 \qquad (R_2 = P_2 \equiv 0).$

Recurrence formula: $\begin{cases} \overbrace{f(s+k)}^{f(s) = (s-3)(s-1)} A_k = -g_2(s)\, A_{k-2} & , \quad k \geq 2 \\ f(s+k)\, A_k = 0 & , \quad k = 0, 1 \end{cases}$; where $\boxed{A_0 \neq 0.}$

④ For $s=s_1=3$ we can always find a solution

$\underline{s=s_1=3}$:

$$\left.\begin{array}{ll} k(2+k)\,A_k = A_{k-2}\,, & k\geq 2 \\ A_k = 0\,, & k=1 \end{array}\right\} \to \left\{ \begin{array}{l} A_k = \dfrac{A_{k-2}}{k(k+2)}\,, \quad k\geq 2 \\ A_1 = 0, \quad A_0 : \text{arbitrary} \end{array}\right.$$

So:

$$\left\{ \begin{array}{l} A_2 = \dfrac{A_0}{2\cdot 4} \\ A_3 = 0 \\ A_4 = \dfrac{A_2}{4\cdot 6} = \dfrac{A_0}{2\cdot4\cdot4\cdot6} \\ \vdots \\ A_{2m+1} = 0 \qquad (k:\text{odd}) \\ A_{2m} = \dfrac{A_{2m-2}}{2m\,(2m+2)} \\ \vdots \end{array}\right.$$

Multiply sides of even coefficients

$$\Longrightarrow \quad A_{2m} = \frac{A_0}{2^2(1\cdot2)\cdot 2^2(2\cdot3)\cdots 2^2(m+1)m} = \frac{A_0}{2^{2m}\,[2\cdot3\cdot4\dots m]^2(m+1)} = \frac{A_0}{2^{2m}(m!)^2(m+1)}$$

So,
$$y_1(x) = x^3\sum_{m=0}^{\infty} A_{2m}\,x^{2m} = A_0x^3\sum_{m=0}^{\infty}\frac{x^{2m}}{2^{2m}(m!)^2(m+1)} \equiv A_0\,u_1(x)\,.$$

In order to see if we can find any solution for $s=s_2=1$, <u>check</u> the recurrence formula for $s=s_2=1$ and $\underline{k=2}$ (because $\underline{s_1-s_2=2}$).

$\underline{k=2}$: $\quad \overset{f(3)}{0}\cdot A_2 = 1\cdot A_0 \neq 0 \qquad$: impossible!

Hence, we can find only 1 independent solution by the Frobenius method

⑤ A second independent solution is of the form

$$y_2(x) = C\,\overset{\text{soln. for } s=s_1}{u_1(x)}\,\ln x + \sum_{m=0}^{\infty} B_m\,x^{2m+\overset{s_2}{1}}\,, \qquad C\neq 0\text{: arbitrary.}$$

The general solution will be of the form : $\quad y(x) = \overbrace{A_0\,u_1(x)}^{y_1(x)} + y_2(x)$.

A_0: arbitrary.