DAVID SHIROKOFF: Hi, everyone. So for this problem, we're just going to take a look at computing some eigenvalues and eigenvectors of several matrices. And this is just a review problem for exam number three.

So specifically, we're given a projection matrix which has the form of a a transpose divided by a transpose a, where a is the vector 3 and 4. The second problem is for a rotation matrix Q which is the numbers 0.6, negative 0.8, 0.8, and 0.6. And then the third one is for a reflection matrix which is 2P minus the identity. So I'll let you work these out. And then I'll come back in a second, and I'll fill in my solutions.

Hi, everyone. Welcome back. OK, so for the first problem, we're given a matrix P, which is a projection matrix. And from earlier on in the course, we probably already know that the eigenvalues of a projection matrix are either 0 or 1.

And I'll just recall, how do you know that? Well if x is an eigenvector of P, then it satisfies the equation Px equals lambda x. But for a projection matrix, P squared is equal to P. So if P is a projection, we have P squared equals P.

And specifically, what this means is P squared x is equal to lambda x. So we have P acting on P of x is equal to lambda x. And on the left hand side, Px is going to give me a lambda x. Px again will give me a lambda x. So we get lambda squared x equals lambda x.

And if I bring everything to the left hand side, I get lambda times lambda minus 1 x equals 0. And because x is not a zero vector, what that means is lambda has to be either 0 or 1. So this is just a quick proof that the eigenvalue of a projection matrix is either 0 or 1.

So we already know that P is going to have eigenvalues of 0 or 1. Now specifically, how do I identify which eigenvectors correspond to 0 and which eigenvectors correspond to 1? Well, in this case, P has a specific form, which is a times a transpose divided by a transpose a.

So I'll just write out explicitly what this is. So a transpose a, 1 divided by a transpose a, is going to be 9 plus 16 on the denominator. Then we're going to have 3 and 4 and 3 and 4.

Now when we have a matrix of this form, it's always going to be the case that the vector a is going to be an eigenvector with eigenvalue 1. So let's check. What is P acting on a?

Well, we end up with the matrix P is 1/25 3, 4, 3, 4. This is the matrix P. And if we acted on the vector 3, 4, notice how this piece right here, we can multiply out. This is going to be a transpose, and this is going to be a. And if we multiply these two pieces out, we get 25, which is exactly the denominator a transpose a.

So at the end of the day, we get 3,4. Because the 25 divides out with the 25. Now note that this is exactly what we started with. This is exactly a. So note here that the vector a corresponds to an eigenvalue of 1.

Meanwhile, for an eigenvalue of 0, well, it always turns out to be the case that if I take any vector perpendicular to a, P acting on that vector is going to be 0. So what's a vector, which I'll call b, that's perpendicular to a?

Well, note that a is just a two by two vector. So that means there's only going to be one direction that's perpendicular to a. Now just by eyeballing it, I can see that a vector that's going to be perpendicular to a is negative 4 and 3. So let's quickly check that this is an eigenvector of P with eigenvalue 0. So what we need to show is that P acting on this vector, b, is 0.

So P acting on b is going to be 1/25. It's going to be 3, 4, 3, 4, multiplied by negative 4, 3. And note how when I multiply out this row on this column, I get negative 3 times 4 plus 3 times 4, which is going to be 0. OK? So this shows that this vector b has an eigenvalue of 0 because note that we can write this as 0b.

OK. For the second part, Q, what are the eigenvectors and eigenvalues of this matrix, Q? Well, Q is a rotation matrix. So I'll just write out Q again, 0.6, negative 0.8, 0.8, 0.6.

So note that we can identify the diagonal elements with a cosine of some angle theta. And we can associate the off diagonal parts as sine theta and negative sine theta. And the reason we can do that is because 0.6 squared plus 0.8 squared is 1. So this is a rotation matrix.

Now to work out the eigenvalues, I take a look at the characteristic equation. So this is going to give me, if I take a look at the characteristic equation, it's going to be 0.6 minus lambda, squared. Then we have minus times 0.8 times negative 0.8. So that's going to be plus 0.8 squared. And we want this to be 0.

So if I rewrite this, I get lambda is 0.6 plus or minus 0.8i, where i is the imaginary number. So notice how the eigenvalues come in complex conjugate pairs. And this is always the case when we have a real matrix.

So we can find, first off, just the eigenvalue that corresponds to 0.6 plus 0.8i. And then at the end, we'll be able to find the second eigenvector by just taking the complex conjugate of the first one. So let's compute Q minus lambda i.

And if we have this acting on some eigenvector, u, we want this to be 0. Now Q minus lambda i is going to be, for the case lambda is 0.6 plus 0.8i, this is going to give me a quantity of minus 0.8i, minus 0.8, 0.8, and minus 0.8i. And I'm going to write down components of u, which are u1 and u2. And we want this to vanish.

And we note that the second row is a constant multiple of the first row. Specifically, if I multiplied this first row through by i, we would get negative i squared, which is just 1. And then the second part would be negative i, so we would just get the second row back, which is good.

So we just need to find u1, u2 that are orthogonal to this first row. And again, just by inspection, I can pick 1 and negative i. So note that that would give me negative 0.8i plus 0.8i, and this vanishes. So this is the eigenvector that corresponds to the eigenvalue lambda 0.6 plus 0.8i.

In the meantime, if I take the second eigenvalue, which is negative 0.8i, I can take u, which is just the complex conjugate of this u up here. So it'll be 1 plus i. So this concludes the eigenvalues and eigenvectors of this matrix Q.

OK. Now lastly, number three, we're looking at a reflection matrix which has the form 2P minus I, where P is the same matrix that we had in part one. Now at first glance, it looks like we might have to diagonalize this entire matrix. However, note that by shifting 2P by I, we only shift the eigenvalues. And we don't actually change the eigenvectors. So note that this matrix R, which is 2P minus I, it's going to have the same eigenvectors as P. It's just going to have different eigenvalues.

So first off, we're going to have one eigenvector. So the first eigenvector is going to be a. So we have one eigenvector which is a. So we have one eigenvector which is a.

And note that for the vector a, it corresponds to the eigenvalue of 1. So what eigenvalue does this correspond to? This is going to give me a lambda which is 2 times 1 minus 1. So it's 1. So note that a, the vector a, not only has an eigenvalue of 1 for P, but it has an eigenvalue of 1 for R as well.

The second case was b. And remember that b has an eigenvalue of 0 for P. So when we act R acting on b, we'll have 2 times 0 minus 1b. So this is going to give us negative b.

So the eigenvalue for b is going to be negative 1. OK. And this is actually a general case for reflection matrices, is that they typically have eigenvalues of plus 1 or negative 1.

OK, so we've just taken a look at several matrices that come up in practice. We've looked at projection matrices, reflection matrices, and rotation matrices. And we've seen a little bit of the properties of their eigenvalues and eigenvectors. So I'll just conclude here, and good luck on your test.