18.06 Linear Algebra, Fall 2011
Recitation Transcript – An Overview of Key Ideas

MARTINA BALAGOVIC: Welcome. Today's problem actually appeared in a quiz. It appeared in quiz one in fall of 1999 as question four. The problem puts the usual solve the following system upside down by saying we have some matrix and we know that all the solutions to ax equals this vector here, 1, 4, 1, 1, all the solutions to this problem are given by x equals 0, 1, 1 plus any number c times 1, 2, 1. And we're asked to say everything that we can about the columns of the matrix A.

So I'm going to let you pretend that you are on an exam, try to solve it yourself, and then come back and compare your solution with mine.

OK, welcome back. So the first thing that you should think about in this sort of situation is what is the size of a? Well if we want to multiply a with an x, that has three entries, so a should have three columns. Let me call those columns c1, c2, and c3. And when I take some linear combinations of c1, c2 and c3, I'm going to get this vector here, 1, 4, 1, 1. So all the ci's, c1, c2, and c3 are vectors in R4.

Now, if you know about, if you learned about particular solutions and special solutions, then my notation here shouldn't surprise you. I'm going to call this vector here xp, and this vector here xs. And I'm going to use the fact that xp plus c times xs satisfies A times this equals b. I will call this vector b, for any number c.

In particular, what I'm going to conclude is that when c equals 0 we get a times xp equals b. But also that when c equals 1, we get a times xp plus a times xs equals b. Replacing this by b, we get that this implies that a times Axs equals 0.

So in trying to find what are the columns, c1, c2, and c3 of the matrix A, let's look at these two equations. xp satisfies a times xp equals b, and xs satisfies A times xs equals 0. Again, if you know what a particular and special solutions are this shouldn't surprise you. But we also know what xp and xs are, so let's use them to try to calculate c1, c2, and c3.

A times xp equals b means that the linear combination of c1, c2, and c3 encoded in the vector xp, which is 0, 1, 1, gives the vector b. So c1, c2, c3 times 0, 1, 1 gives us 1, 4, 1, 1. In other words, c2 plus c3 equal b.

Let's turn our attention to A times xs equals 0. This says that c1, c2, c3 times xs was defined to be 0, 2, 1 equals 0. In other words, 2 times c2 plus c3 equals 0.

Now solving this system where the unknowns are vectors but it's still just a linear system, we can see, for example, from the second equation that c3 equals minus 2c2. And plugging it back into the original equation getting c2 minus 2c2 equals b from which it follows that c2 is equal to minus b, and that c3 is equal to 2 times b.

So from this tiny amount of information, we just knew the solutions to this one particular equation involving A, we got the second column of A and the third column of A completely explicitly calculated.

Now, what can we say about the first column? I said before that all the solutions of ax equals b are of the form a particular solution plus some number times a special solution. And the information that we have is that there's just one number here. So we said everything, once we remove this vector here, everything that we get here will satisfy A times x equals 0.

And the fact that everything that satisfies A times x equals 0 is a multiple of this one vector that was given to us means that the null space of A has dimension one. There is just one special solution. So dimension of the null space of A is 1. So rank of A is the number of columns minus this dimension of null space, and it's equal to 2.

As rank of A is equal to 2, the number of linearly independent columns needs to be 2 as well. So the only thing that we can say at this point is if the first column was also a multiple of b, as the second and the third are, then the rank would be smaller than 2. So that is the only thing that cannot happen. So c1 is not a multiple of b. Not any multiple, including not a 0 multiple.

And that's pretty much everything we can say. Yes, if it was some other multiple of it, then we would be able to find some other vector here and we would have two parameters. But it's not, and this is everything that we can say about it.

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