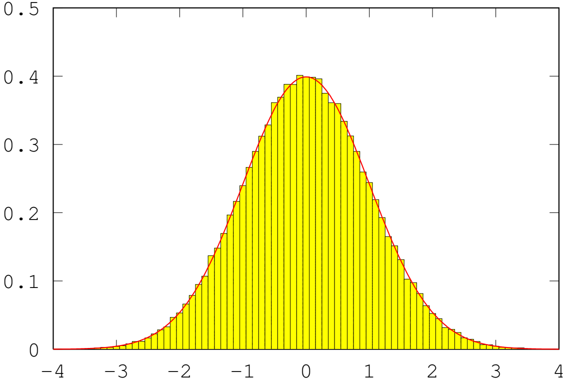
Continuous Expectation and Variance, the Law of Large Numbers, and the Central Limit Theorem

18.05 Spring 2014

Jeremy Orloff and Jonathan Bloom



Expected value

Expected value: measure of location, central tendency

X continuous with range $[a, b]$ and pdf $f(x)$:

$$E(X) = \int_a^b x f(x)\, dx.$$

X discrete with values $x_1, \ldots, x_n$ and pmf $p(x_i)$:

$$E(X) = \sum_{i=1}^{n} x_i p(x_i).$$

View these as essentially the same formulas.

Variance and standard deviation

Standard deviation: measure of spread, scale

For *any* random variable X with mean μ

$$\mathrm{Var}(X) = E((X-\mu)^2), \qquad \sigma = \sqrt{\mathrm{Var}(X)}$$

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Properties

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1. $E(X+Y) = E(X) + E(Y)$.
2. $E(aX+b) = aE(X) + b$.

1. If X and Y are independent then $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.
2. $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$.
3. $\mathrm{Var}(X) = E(X^2) - E(X)^2$.

Board question

The random variable X has range [0,1] and pdf cx^2.

a) Find c.

b) Find the mean, variance and standard deviation of X.

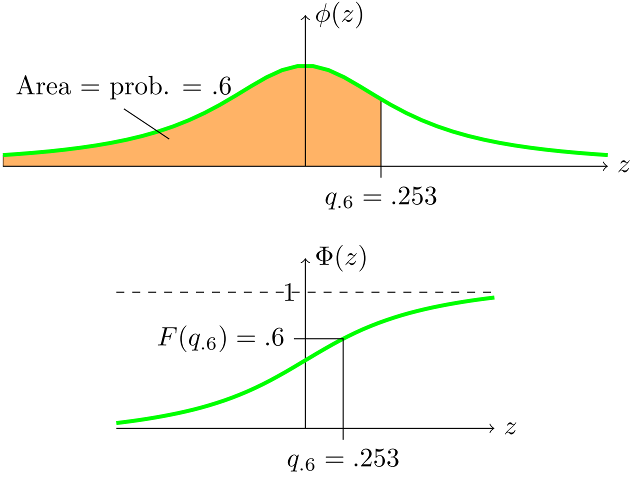
c) Find the median value of X.

d) Suppose $X_1, \ldots X_{16}$ are independent identically-distributed copies of X. Let $\overline{X}$ be their average. What is the standard deviation of $\overline{X}$?

e) Suppose $Y = X^4$. Find the pdf of Y.

Quantiles

Quantiles give a measure of **location.**

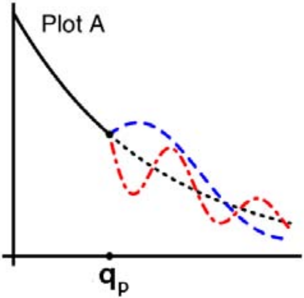


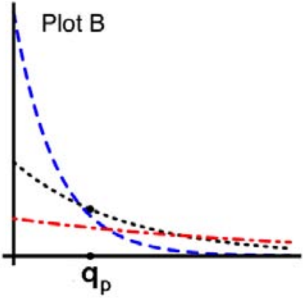
$$q_{.6}\text{: } \textit{left tail area} = .6 \iff F(q_{.6}) = .6$$

Concept question

In each of the plots some densities are shown. The median of the black plot is always at q_p. In each plot, which density has the greatest median?

1. Black
2. Red
3. Blue
4. All the same
5. Impossible to tell





Law of Large Numbers (LoLN)

Informally: An average of many measurements is more accurate than a single measurement.

Formally: Let $X_1, X_2, \ldots$ be i.i.d. random variables all with mean μ and standard deviation σ.
Let

$$\overline{X}_n = \frac{X_1 + X_2 + \ldots + X_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Then for any (small number) a, we have

$$\lim_{n\to\infty} P(|\overline{X}_n - \mu| < a) = 1.$$

Concept Question: Desperation

- You have \$100. You need \$1000 by tomorrow morning.
- Your only way to get it is to gamble.
- If you bet \$k, you either win \$k with probability p or lose \$k with probability $1 - p$.

Maximal strategy: Bet as much as you can, up to what you need, each time.
Minimal strategy: Make a small bet, say \$5, each time.

1. If $p = .45$, which is the better strategy?

A. Maximal   B. Minimal

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   A. Maximal B. Minimal

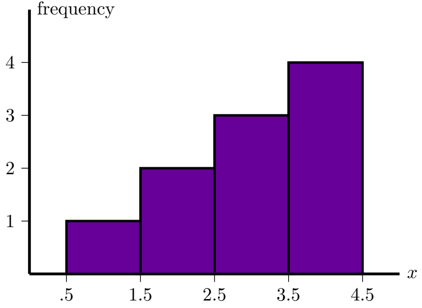
2. If $p = .8$, which is the better strategy?
   A. Maximal B. Minimal

Histograms

Made by 'binning' data.

Frequency: *height* of bar over bin = number of data points in bin.

Density: *area* of bar is the fraction of all data points that lie in the bin. So, total area is 1.



Board question

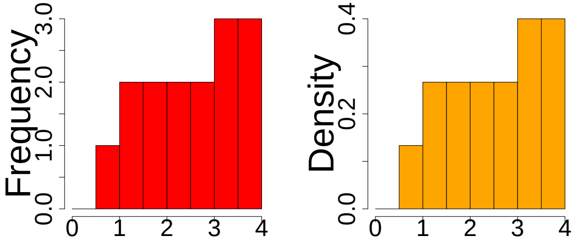
1. Make a both a frequency and density histogram from the data below.

Use bins of width 0.5 starting at 0. The bins should be right closed.

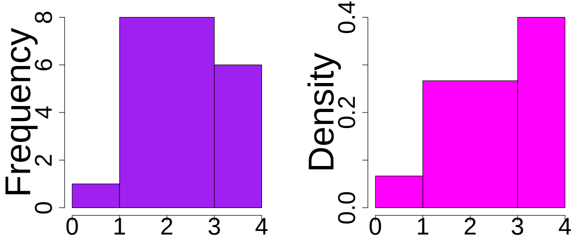
1	1.2	1.3	1.6	1.6
2.1	2.2	2.6	2.7	3.1
3.2	3.4	3.8	3.9	3.9

2. Same question using unequal width bins with edges 0, 1, 3, 4.

Solution



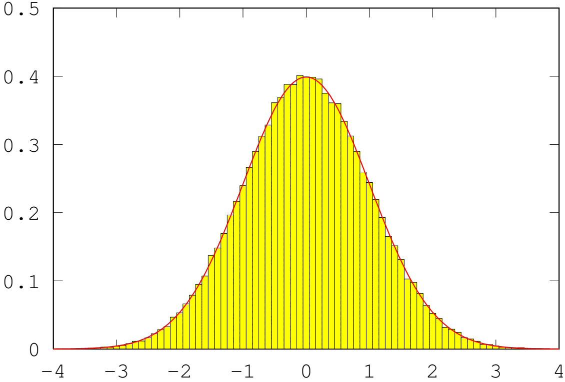
Histograms with equal width bins



Histograms with unequal width bins

LoLN and histograms

LoLN implies density histogram converges to pdf:



Histogram with bin width .1 showing 100000 draws from a standard normal distribution. Standard normal pdf is overlaid in red.

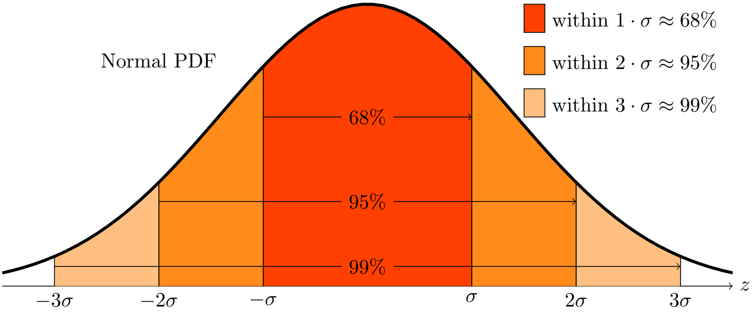
Standardization

Random variable X with mean μ and standard deviation σ.

Standardization: $Z = \dfrac{X - \mu}{\sigma}$.

- Z has mean 0 and standard deviation 1.
- Standardizing any normal random variable produces the standard normal.
- If $X \approx$ normal then standardized $X \approx$ stand. normal.

Concept Question: Standard Normal



1. $P(-1 < Z < 1)$ is
   a) .025 b) .16 c) .68 d) .84 e) .95
2. $P(Z > 2)$
   a) .025 b) .16 c) .68 d) .84 e) .95

Central Limit Theorem

Setting: X_1, X_2, ... i.i.d. with mean μ and standard dev. σ.

For each n:

$$\overline{X}_n = \frac{1}{n}(X_1 + X_2 + \ldots + X_n)$$
$$S_n = X_1 + X_2 + \ldots + X_n.$$

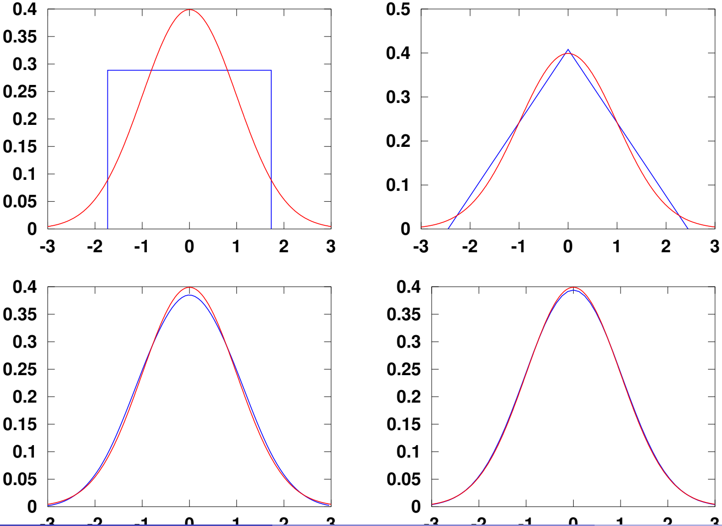
Conclusion: For large n:

$$\overline{X}_n \approx \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$
$$S_n \approx \mathrm{N}\left(n\mu, n\sigma^2\right)$$

Standardized S_n or $\overline{X}_n \approx \mathrm{N}(0, 1)$

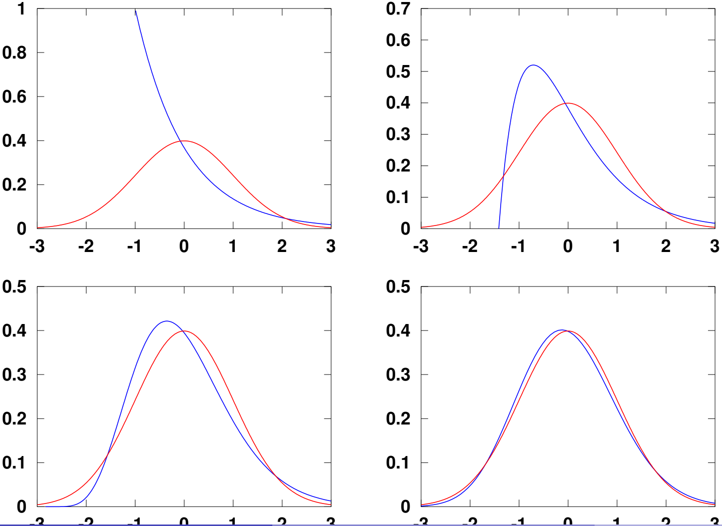
CLT: pictures

Standardized average of n i.i.d. uniform random variables with $n = 1, 2, 4, 12$.



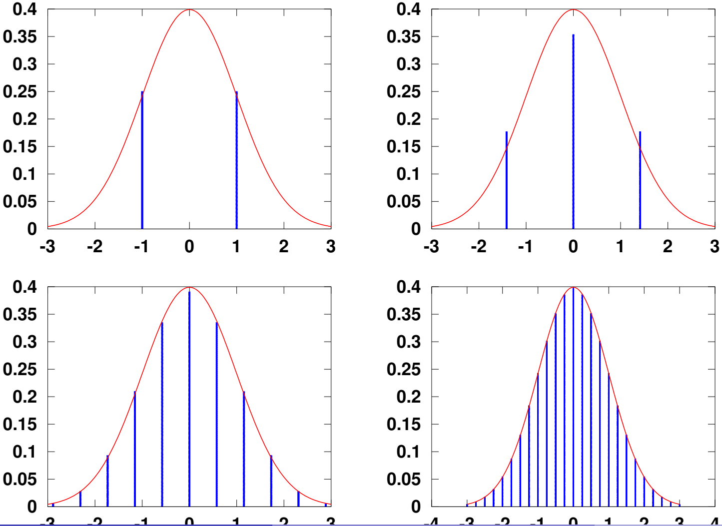
CLT: pictures 2

The standardized average of n i.i.d. exponential random variables with $n = 1, 2, 8, 64$.



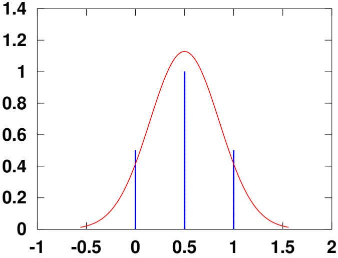
CLT: pictures 3

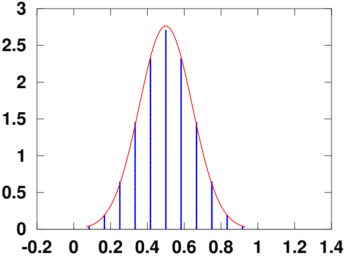
The standardized average of n i.i.d. Bernoulli(.5) random variables with $n = 1, 2, 12, 64$.



CLT: pictures 4

The (non-standardized) average of n Bernoulli(.5) random variables, with $n = 4, 12, 64$. (Spikier.)





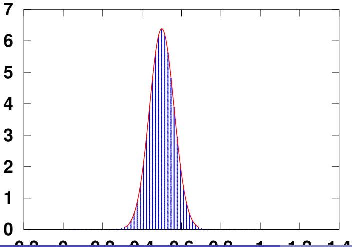


Table Question: Sampling from the standard normal distribution

As a table, produce a single random sample from (an approximate) standard normal distribution.

The table is allowed nine rolls of the 10-sided die.

Note: $\mu = 5.5$ and $\sigma^2 = 8.25$ for a single 10-sided die.

Hint: CLT is about averages.

Board Question: CLT

1. Carefully write the statement of the central limit theorem.

2. To head the newly formed US Dept. of Statistics, suppose that 50% of the population supports Erika, 25% supports Ruthi, and the remaining 25% is split evenly between Peter, Jon and Jerry.

A poll asks 400 random people who they support. What is the probability that at least 55% of those polled prefer Erika?

3. What is the probability that less than 20% of those polled prefer Ruthi?

18.05 Introduction to Probability and Statistics
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