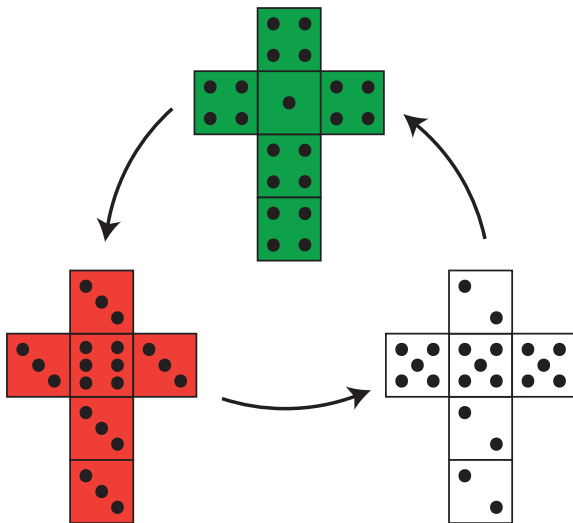


Probability: Terminology and Examples
18.05 Spring 2014
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Discussion Boards

The screenshot shows the MITx courseware interface for the course "MITx: 18.05x Introduction to Probability and Statistics". The navigation menu includes "Courseware", "Course Info", "Discussion", "Progress", and "Instructor". The "Discussion" tab is highlighted with a red circle. On the left sidebar, the "Problem set 1" section is selected. The main content area displays "PROBLEM SET 1" with a link to "See Problem set 1". At the bottom of the content area, there are two buttons: "Show Discussion" and "New Post", both of which are circled in red.

MITx MITx: 18.05x Introduction to Probability and Statistics

Courseware Course Info **Discussion** Progress Instructor

Calendar and Information

Week 1 (2/3-2/7)

Class 1 (T 2/4)
Introduction; counting

Class 1 slides

Class 2 (R 2/6)
Probability basics

Problem set 1

PROBLEM SET 1

See [Problem set 1](#)

Show Discussion

New Post

Board Question

Deck of 52 cards

- 13 *ranks*: 2, 3, ..., 9, 10, J, Q, K, A
- 4 *suits*: ♥, ♠, ♦, ♣,

Poker hands

- Consists of 5 cards
- A *one-pair* hand consists of two cards having one rank and the remaining three cards having three other rank
- Example: {2♥, 2♠, 5♥, 8♣, K♦}

Question

a) How many different 5 card hands have exactly one pair?

Hint: practice with how many 2 card hands have exactly one pair.

Hint for hint: use the rule of product.

b) What is the probability of getting a one pair poker hand?

Answer to board question

We can do this two ways as combinations or permutations. The keys are:

1. be consistent
2. break the problem into a sequence of actions and use the rule of product.

Note, there are many ways to organize this. We will break it into very small steps in order to make the process clear.

Combinations approach

a) Count the number of one-pair hands, where the order they are dealt doesn't matter.

Action 1. Choose the rank of the pair: 13 different ranks, choosing 1, so $\binom{13}{1}$ ways to do this.

Action 2. Choose 2 cards from this rank: 4 cards in a rank, choosing 2, so $\binom{4}{2}$ ways to do this.

Action 3. Choose the 3 cards of different ranks: 12 remaining ranks, so $\binom{12}{3}$ ways to do this.

(Continued on next slide.)

Combination solution continued

Action 4. Choose 1 card from each of these ranks: 4 cards in *each* rank so $\binom{4}{1}^3$ ways to do this.

answer: (Using the rule of product.)

$$\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1098240$$

b) To compute the probability we have to stay consistent and count combinations. To make a 5 card hand we *choose* 5 cards out of 52, so there are

$$\binom{52}{5} = 2598960$$

possible hands. Since each hand is equally likely the probability of a one-pair hand is

$$1098240/2598960 = 0.42257.$$

On the next slide we redo this problem using permutations.

Permutation approach

This approach is a little trickier. We include it to show that there is usually more than one way to count something.

a) Count the number of one-pair hands, where we keep track of the order they are dealt.

Action 1. (This one is tricky.) Choose the positions in the hand that will hold the pair: 5 different positions, so $\binom{5}{2}$ ways to do this.

Action 2. Put a card in the first position of the pair: 52 cards, so 52 ways to do this.

Action 3. Put a card in the second position of the pair: since this has to match the first card, there are only 3 ways to do this.

Action 4. Put a card in the first open slot: this can't match the pair so there are 48 ways to do this.

Action 5. Put a card in the next open slot: this can't match the pair or the previous card, so there 44 ways to do this.

Action 6. Put a card in the last open slot: there are 40 ways to do this.

(Continued on next slide.)

Permutation approach continued

answer: (Using the rule of product.)

$$\binom{5}{2} \cdot 52 \cdot 3 \cdot 48 \cdot 44 \cdot 40 = 131788800$$

ways to deal a one-pair hand where we keep track of order.

b) There are

$${}_{52}P_5 = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311875200$$

five card hands where order is important.

Thus, the probability of a one-pair hand is

$$131788800/311875200 = 0.42257.$$

(Both approaches give the same answer.)

Clicker Test

Set your clicker channel to 41.

Do you have your clicker with you?

No = 0

Yes = 1

Probability Cast

Introduced so far

- Experiment: a repeatable procedure
- Sample space: set of all possible outcomes S (or Ω).
- Event: a subset of the sample space.
- Probability function, $P(\omega)$: gives the probability for each outcome $\omega \in S$
 1. Probability is between 0 and 1
 2. Total probability of all possible outcomes is 1.

Example (from the reading)

Experiment: toss a fair coin, report heads or tails.

Sample space: $\Omega = \{H, T\}$.

Probability function: $P(H) = .5$, $P(T) = .5$.

Use tables:

| | | |
|-------------|-----|-----|
| Outcomes | H | T |
| Probability | 1/2 | 1/2 |

(Tables can really help in complicated examples)

Discrete sample space

Discrete = listable

Examples:

$\{a, b, c, d\}$ (finite)

$\{0, 1, 2, \dots\}$ (infinite)

Events

Events are sets:

- Can describe in words
- Can describe in notation
- Can describe with Venn diagrams

Experiment: toss a coin 3 times.

Event:

You get 2 or more heads = { HHH, HHT, HTH, THH }

CQ: Events, sets and words

Experiment: toss a coin 3 times.

Which of following equals the event “exactly two heads”?

$$A = \{THH, HTH, HHT, HHH\}$$

$$B = \{THH, HTH, HHT\}$$

$$C = \{HTH, THH\}$$

- (1) A (2) B (3) C (4) A or B

answer: : 2) B.

The event “exactly two heads” determines a *unique subset*, containing *all* outcomes that have exactly two heads.

CQ: Events, sets and words

Experiment: toss a coin 3 times.

Which of the following describes the event $\{THH, HTH, HHT\}$?

- (1) “exactly one head”
- (2) “exactly one tail”
- (3) “at most one tail”
- (4) none of the above

answer: (2) “exactly one tail”

Notice that the same event $E \subset \Omega$ may be described in words in multiple ways (“exactly 2 heads” and “exactly 1 tail”).

CQ: Events, sets and words

Experiment: toss a coin 3 times.

The events “exactly 2 heads” and “exactly 2 tails” are disjoint.

(1) True (2) False

answer: True: $\{THH, HTH, HHT\} \cap \{TTH, THT, HTT\} = \emptyset$.

CQ: Events, sets and words

Experiment: toss a coin 3 times.

The event “at least 2 heads” implies the event “exactly two heads”.

(1) True (2) False

False. It's the other way around:

$\{THH, HTH, HHT\} \subset \{THH, HTH, HHT, HHH\}$.

Probability rules in mathematical notation

Sample space: $S = \{\omega_1, \omega_2, \dots, \omega_n\}$

Outcome: $\omega \in S$

Probability between 0 and 1: $0 \leq P(\omega) \leq 1$

Total probability is 1: $\sum_{j=1}^n P(\omega_j) = 1, \quad \sum_{\omega \in S} P(\omega) = 1$

Event A : $P(A) = \sum_{\omega \in A} P(\omega)$

Probability and set operations on events

Events A, L, R

Rule 1. Complements. $P(A^c) = 1 - P(A)$.

Rule 2. Disjoint events.

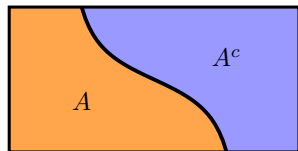
If L and R are disjoint then

$$P(L \cup R) = P(L) + P(R).$$

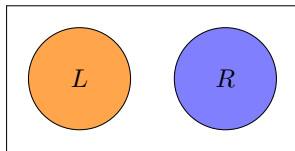
Rule 3. Inclusion-exclusion principle.

Any L and R :

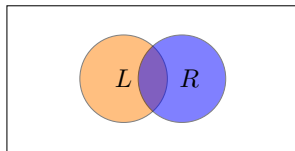
$$P(L \cup R) = P(L) + P(R) - P(L \cap R)$$



$\Omega = A \cup A^c$, no overlap



$L \cup R$, no overlap



$L \cup R$, overlap = $L \cap R$

Concept question

- Class has 50 students
- 20 male (M), 25 brown-eyed (B)

For a randomly chosen student what is the range of possible values for $p = P(M \cup B)$?

- (a) $p \leq .4$
- (b) $.4 \leq p \leq .5$
- (c) $.4 \leq p \leq .9$
- (d) $.5 \leq p \leq .9$
- (e) $.5 \leq p$

answer: (d) $.5 \leq p \leq .9$

Explanation on next slide.

Solution to CQ

The easy way to answer this is that $A \cup B$ has a minimum of 25 members (when all males are brown-eyed) and a maximum of 45 members (when no males have brown-eyes). So, the probability ranges from .5 to .9

Thinking about it in terms of the inclusion-exclusion principle we have

$$P(M \cup B) = P(M) + P(B) - P(M \cap B) = .9 - P(M \cap B).$$

So the maximum possible value of $P(M \cup B)$ happens if M and B are disjoint, so $P(M \cap B) = 0$. The minimum happens when $M \subset B$, so $P(M \cap B) = P(M) = .4$.

Table Question

Experiment:

- 1) Roll your 20-sided die.
- 2) Check if all rolls at your table are distinct.

Repeat the experiment five times and record the results.

Table Question

Experiment:

- 1) Roll your 20-sided die.
- 2) Check if all rolls at your table are distinct.

Repeat the experiment five times and record the results.

For this experiment, how would you define the sample space, probability function, and event?

Compute the exact probability that all rolls are distinct.

answer: $1 - (((20!)/(11!))/(20^9)) = 0.881$

The explanation is on the next frame.

Board Question Solution

For the sample space S we will take all sequences of 9 numbers between 1 and 20. We find the size of S using the rule of product. There are 20 ways to choose the first number in the sequence, followed by 20 ways to choose the second, etc. Thus, $|S| = 20^9$.

It is sometimes easier to calculate the probability of an event indirectly by calculating the probability of the complement and using the formula

$$P(A) = 1 - P(A^c).$$

In our case, A is the event 'there is a match', so A^c is the event 'there is no match'. We can use the rule of product to compute $|A^c|$ as follows. There are 20 ways to choose the first number, then 19 ways to choose the second, etc. down to 12 ways to choose the ninth number. Thus, we have

$$|A^c| = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12$$

Putting this all together

$$P(A) = 1 - P(A^c) = 1 - \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{20^9} = .881$$

Concept Question

Lucky Larry has a coin that you're quite sure is not fair.

- He will flip the coin twice
- It's your job to bet whether the outcomes will be the same (HH, TT) or different (HT, TH).

Which should you choose?

1. Same
2. Different
3. It doesn't matter, same and different are equally likely

Board Question

Lucky Larry has a coin that you're quite sure is not fair.

- He will flip the coin twice
- It's your job to bet whether the outcomes will be the same (HH, TT) or different (HT, TH).

Which should you choose?

1. Same 2. Different 3. Doesn't matter

Question: Let p be the probability of heads and use probability to answer the question.

(If you don't see the symbolic algebra try $p = .2$, $p = .5$)

Solution

answer: 1. Same (same is more likely than different)

The key bit of arithmetic is that if $a \neq b$ then

$$(a - b)^2 > 0 \Leftrightarrow a^2 + b^2 > 2ab.$$

To keep the notation cleaner, let's use $P(T) = (1 - p) = q$. Since the flips are independent (we'll discuss this next week) the probabilities multiply. This gives the following 2×2 table.

| | | | |
|------------|---|-------------|-------|
| | | second flip | |
| | | H | T |
| first flip | H | p^2 | pq |
| | T | qp | q^2 |

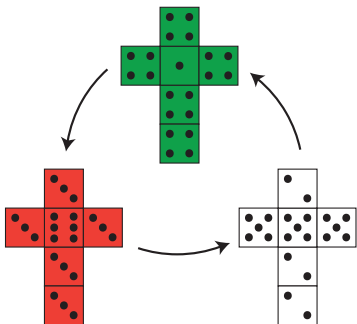
So, $P(\text{same}) = p^2 + q^2$ and $P(\text{diff}) = 2pq$. Since the coin is unfair we know $p \neq q$. Now we use our key bit of arithmetic to say

$$p^2 + q^2 > 2pq \Rightarrow P(\text{same}) > P(\text{different}).$$

QED

Jon's dice

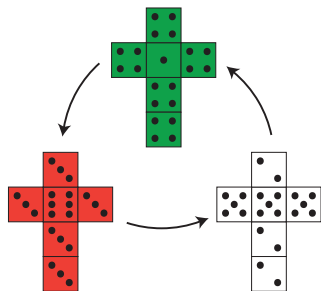
Jon has three six-sided dice with unusual numbering.



A game consists of two players each choosing a die. They roll once and the highest number wins.

Which die would you choose?

Board Question



Compute the exact probability of red beating white.

- 1) For red and white dice make the probability table.
- 2) Make a prob. table for the product sample space of red and white.
- 3) What is the probability that red beats white?

answer: $7/12$

Answer to board question

For each die we have a probability table

| | | | | | |
|-------------|---------|-----|-------------|-----------|-----|
| | Red die | | | White die | |
| Outcomes | 3 | 6 | Outcomes | 2 | 5 |
| Probability | 5/6 | 1/6 | Probability | 1/2 | 1/2 |

For both together we get a 2×2 probability table

| | | | |
|-----|---|-------|------|
| | | White | |
| | | 2 | 5 |
| Red | 3 | 5/12 | 5/12 |
| | 6 | 1/12 | 1/12 |

The red table entries are those where red beats white. Totalling the probability we get $P(\text{red beats white}) = 7/12$.

Concept Question

We saw red is better than white.

We'll tell you that white is better than green.

So red is better than white is better than green.

Is red better than green?

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18.05 Introduction to Probability and Statistics

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