18.04 Ancient History #2

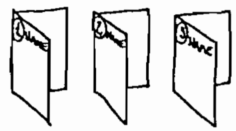
Fri 24 Oct 03

18.04 Exam #2

Friday, April 4, 1997

CLOSED BOOK

Once again, please struggle with Problems 1, 2 and 3 on separate sheets of paper ...



and also indicate your RECITATION: M2 M3 Tu11 Tu1 Tu2. Thanks!

① (a) State Cauchy's integral formula and any restrictions

(b) Specialize it to a circle of radius a centered on z_o

(c) Hence show that $|f(z_o)| \le \max |f(z_o + ae^{i\theta})|$

(d) Finish this proof of the maximum modulus theorem

② (a) Obtain the first four non-zero terms of the Laurent expansion that validly represents the function

$$f(z) = \frac{e^z}{z^3\,(z^2+4)}$$

in the region $0 < |z| < 2$.

(b) Use this valuable information somehow to compute or estimate the integral

$$\oint f(z)\,dz ,$$

where the path is the unit circle $|z| = 1$, to be traversed exactly once in the counterclockwise sense.

③ Use residue calculus to evaluate

$$I(a) = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+a^2)}$$

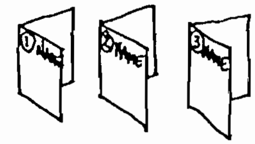
for any real, positive value of the constant a , excluding only unity. And for a full score also confirm that your formula for $I(a)$ behaves correctly in the limit as $a \to 1$, a special case that you are welcome to "regurgitate" either from residue theory reapplied to that second-order pole, or else via the old trick $x = \tan\beta$ from calculus.

18.04 Exam #2 /4

Wednesday, March 20, 1996

CLOSED BOOK

Again, please ...



① Express

$$S(\theta) = \sum_{k=-5}^{5} e^{ik\theta}$$

as the ratio of two sines.

HINT: Employ $e^{\pm i\theta/2}$ somehow, to your great advantage.

② Let C be the unit square with vertices at the points $z = 0,\ 1,\ 1+i$ and i . Once around this path, in the counterclockwise sense, please integrate

$$\oint \exp \bar{z}\,dz ,$$

where $\bar{z}$ denotes the usual complex conjugate of z .

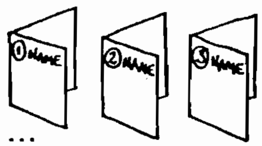
③ Once around the ellipse $4x^2 + y^2 = 9$ nicknamed 'E', integrate

$$\oint_E \frac{\sin z}{z^2+4}\,dz .$$

18.04 Exam #2 Wednesday, April 3, 1991

CLOSED BOOK

As in February,
please struggle with
Problems 1, 2 and 3 on
separate sheets of paper ...



(1) For each of the following, determine whether the statement made is always true or sometimes false:

(a) If f and g have a pole at z_0, then $f + g$ has a pole at z_0.
(b) If f has an essential singularity at z_0 and g has a pole at z_0, then $f + g$ has an essential singularity at z_0.
(c) If $f(z)$ has a pole of order m at $z = 0$, then $f(z^2)$ has a pole of order $2m$ at $z = 0$.
(d) If f has a pole at z_0 and g has an essential singularity at z_0, then the product $f \cdot g$ has a pole at z_0.
(e) If f has a zero of order m at z_0 and g has a pole of order n, $n \le m$, at z_0, then the product $f \cdot g$ has a removable singularity at z_0.

And remember that a few words of explanation, plus even a counterexample or two when appropriate, will enhance the impression that the writer is immensely wise!

(2) Evaluate the real integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 + x^2 + 1}$.

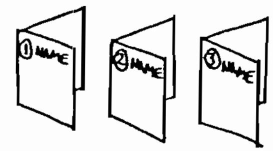
(3) a) Find the residue of $\dfrac{e^{iz}}{(z^2+1)^2}$ at $z = i$.

b) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx$.

18.04 Exam #2 Friday, November 6, 1987

CLOSED BOOK

As usual, please ...



(1) a) Sum the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1+i}{3}\right)^n$.

b) Sum the series $\displaystyle\sum_{n=1}^{\infty} n z^n$ for all $|z| < 1$.

(2) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx$.

(3) Evaluate $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{x^2+4}\, dx$.

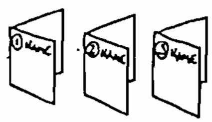
18.04 Modern History #2

Fri 24 Oct 03

18.04 Exam #2 Friday, November 3, 2000

CLOSED BOOK ... and NO calculators

Again please ...



(1) (a) Determine at least the coefficients a_1, a_2, a_3, a_4 needed in this Taylor series:

$$\frac{1}{1+z+z^4} = 1 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots$$

(b) Discuss why we can be quite certain that this series would converge at least out to $|z| = 2/3$ if all the later coefficients were sleuthed out and used as well.

(2) Employ our friend e^{ix} to evaluate neatly and efficiently:

(a) the integral $\int_0^{2\pi} \cos^6 x \, dx$

(b) the sum $\sum_{n=0}^{\infty} \frac{\cos(nx)}{2^n}$

(3) Use residue calculus to evaluate

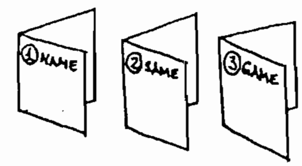
$$\int_0^{2\pi} \frac{d\theta}{(5 - 4\cos\theta)^2} .$$

18.04 Exam #2 Friday, April 2, 1999

CLOSED BOOK ... and NO calculators

Once again, please struggle with these problems on separate sheets of paper ... named and numbered. Thanks.

REWARD: No proofs this time!



(1) Evaluate the integral $\oint_C \tan z \, dz$, assuming C to be the circle $|z| = 2$ traversed once, counterclockwise.

(2) (a) Expand $\frac{z}{(z-5)^2}$ into a Laurent series about $z = 5$.

(b) Obtain the first THREE non-zero terms of the Taylor series for that same function, now centered on $z = 0$.

(c) Obtain the first THREE non-zero terms of its Laurent series centered on $z = 0$ and valid at great distance.

(3) Use residue calculus to evaluate $\int_0^{2\pi} \frac{\cos^2\theta}{5 - 4\cos\theta} \, d\theta$.

① Our function $\tan z = \frac{\sin z}{\cos z}$ is plainly analytic everywhere, *except* at locations like $z = \pm\pi/2$ at which $\cos z$ vanishes in its denominator. Two of those singularities lie inside our circle $|z| = 2$. Near the "trouble spot" $z_1 = +\frac{\pi}{2}$: $\tan z = \frac{1}{z-\frac{\pi}{2}} \cdot \left[\sin z \cdot \frac{z-\frac{\pi}{2}}{\cos z}\right]$, where the part $[\ ] \to 1 \cdot \frac{1}{-1} = -1$ as $z \to \frac{\pi}{2}$.

Hence circling that spot in a ↺ sense will yield $2\pi i \cdot (-1)$ from CIF or residue calculus alike. Near $z_2 = -\frac{\pi}{2}$, on the other hand, $\tan z = \frac{1}{z+\frac{\pi}{2}} \cdot \left[\sin z \cdot \frac{z+\frac{\pi}{2}}{\cos z}\right]$, and now $[\ ] \to -1 \cdot \frac{1}{+1} = -1$ (again!) thereabouts, again via l'Hopital. Hence those two "residues" are *both* -1 rather than opposites and $\boxed{\oint_{|z|=2} \tan z\, dz = -4\pi i}$

② (a) Our $\frac{z}{(z-5)^2} = \frac{5+(z-5)}{(z-5)^2} = \boxed{\frac{5}{(z-5)^2} + \frac{1}{z-5}}$ ← DONE!!

(b) $\ldots = \frac{z}{25} \cdot \left(\frac{1}{1-\frac{z}{5}}\right)^2 = \frac{z}{25}\left(1 + \frac{z}{5} + \frac{z^2}{25} + \ldots\right)^2 = \boxed{\frac{z}{25} + \frac{2z^2}{125} + \frac{3z^3}{625} + \ldots}$

(c) $\ldots = \frac{1}{z} \cdot \left(1 - \frac{5}{z}\right)^{-2} = \frac{1}{z}\left(1 + \frac{5}{z} + \frac{25}{z^2} + \ldots\right)^2 = \boxed{\frac{1}{z} + \frac{10}{z^2} + \frac{75}{z^3} + \ldots}$

And surely these 3 results should be *weighted* like $2+4+4 = 10$ pts.

③ Apart from a trivial sign change in the denominator, and $\cos^2\theta$ replacing $\sin^2\theta$ "upstairs", this problem was deliberately chosen to duplicate the worked-out *Example 1* from p.253 of our text!

Much like there, our

$$I \equiv \int_0^{2\pi} \frac{\cos^2\theta \cdot d\theta}{5 - 4\cos\theta} = \oint_{|z|=1} \frac{(z+\frac{1}{z})^2/4}{5 - 2(z+\frac{1}{z})} \frac{dz}{iz} = -\frac{1}{8i} \oint \frac{(z^2+1)^2}{z^2(z-\frac{1}{2})(z-2)} dz$$

has a relevant simple pole at $z = \frac{1}{2}$ and a pole of 2nd order at $z=0$. (The other simple pole, at $z=2$, lies outside our unit circle, and thus contributes nothing to this integral.) Those relevant residues work out as $Res(\frac{1}{2}) = \frac{(\frac{1}{4}+1)^2}{\frac{1}{4}(\frac{1}{2}-2)} = -\frac{25}{6}$ and as $Res(0) = +\frac{5}{2}$ … from $\frac{d}{dz}\left[\frac{(z^2+1)^2}{z^2 - \frac{5}{2}z + 1}\right]_{z=0}$. Hence our $I = -\frac{2\pi i}{8i}\left(-\frac{25}{6} + \frac{5}{2}\right) = \boxed{\frac{5\pi}{12}}$.

① (a) Since $\frac{1}{1+\varepsilon} = 1 - \varepsilon + \varepsilon^2 - \varepsilon^3 + \ldots$, our $\frac{1}{1+(z+z^4)}$ similarly equals $1 - (z+z^4) + (z+z^4)^2 - (z+z^4)^3 + \ldots = \boxed{1 - z + z^2 - z^3 + 0z^4 + \ldots}$

(b) And the TS convergence theorem of course guarantees that such a series must converge for all $|z|$ less than the cx distance to the nearest zero of our denominator $1 + z + z^4$, because that is the only way that the $f(z)$ in question can have a singularity. Yet from three complex numbers of modulus 1, $z/3$ and $(z/3)^4 = 16/81 < 1/3$ we cannot yet make a sum that adds up to zero, can we?

② (a) $\cos^6 x = \frac{1}{2^6}(e^{ix} + e^{-ix})^6 = \frac{1}{64}(e^{6ix} + 6e^{4ix} + 15e^{2ix} + 20 + \ldots)$ via Euler and $(a+b)^6$. And by now you know — or *ought* to know! — that $\int_0^{2\pi} e^{6ix} dx = \int_0^{2\pi} e^{4ix} dx = \ldots = 0$, leaving us only that $20/64$ as a "net profit", or $\boxed{\int_0^{2\pi} \cos^6 x\, dx = \frac{5\pi}{8}}$

(b) Our sum $1 + \frac{1}{2}\cos x + \frac{1}{4}\cos 2x + \ldots$ likewise collapses via e^{ix} into two ~~simple~~ *geometric* sums $\frac{1}{2}\left[1 + \frac{e^{ix}}{2} + \left(\frac{e^{ix}}{2}\right)^2 + \ldots\right] + \frac{1}{2}\left[1 + \frac{e^{-ix}}{2} + \left(\frac{e^{-ix}}{2}\right)^2 + \ldots\right]$, which work out as $\frac{1}{2}\left[1 - \frac{e^{ix}}{2}\right]^{-1} + \frac{1}{2}\left[1 - \frac{e^{-ix}}{2}\right]^{-1} = \frac{1}{2 - e^{ix}} + \frac{1}{2 - e^{-ix}}$, meaning that our $\boxed{\sum = \frac{4 - 2\cos x}{5 - 4\cos x}}$ … $= 2$ when $x = 0$ ✓
$= 4/5$ when $x = \frac{\pi}{2}$ ✓
$= 2/3$ when $x = \pi$ ✓

③ Via the usual $z = e^{i\theta}$ and $dz = ie^{i\theta} d\theta = iz\, d\theta$, our integral

$$\int_0^{2\pi} \frac{d\theta}{(5 - 4\cos\theta)^2} = \oint \frac{\frac{dz}{iz}}{(5 - 2z - \frac{2}{z})^2} = \frac{1}{4i} \oint \frac{z\, dz}{(z^2 - \frac{5}{2}z + 1)^2},$$

now of course to be done around the unit circle $|z| = 1$. Here our $f(z) = \frac{z}{(z^2 - \frac{5}{2}z + 1)^2} = \frac{z}{(z-2)^2(z-\frac{1}{2})^2}$ has a pole of order 2 at $z = \frac{1}{2}$ inside this circle, with residue $Res = \frac{d}{dz}\left[\frac{z}{(z-2)^2}\right]_{z=\frac{1}{2}} = \frac{20}{27}$.

Hence our $\int \ldots d\theta = \frac{2\pi i}{4i} \cdot \frac{20}{27} = \boxed{\frac{10\pi}{27}}$.

AT