Part I Problems and Solutions

In the next three problems, solve the given DE system $x' = Ax$.
First find the eigenvalues and associated eigenvectors, and from these construct the normal modes and thus the general solution.

Problem 1: *Solve* $\mathbf{x}' = A\mathbf{x}$, *where A is* $\begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$.

Solution: First, we find the eigenvalues: $\begin{vmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix} = 0$ implies $-(3+\lambda(3-\lambda)+8 = 0 \to \lambda^2 - 1 = 0$, so $\lambda = \pm 1$.

Now, we find the eigenvalues. If $\lambda = 1$,

$$\begin{aligned} -4\alpha_1 + 4\alpha_2 &= 0 \\ -2\alpha_1 + 2\alpha_2 &= 0 \end{aligned}$$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and its multiples are solutions and thus eigenvectors.

If $\lambda = -1$, we obtain $-2\alpha_1 + 4\alpha_2 = 0$ (twice), and thus $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector.

Therefore, $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$.

Problem 2: *Solve* $\mathbf{x}' = A\mathbf{x}$ *where A is* $\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$.

Solution: First, we find the eigenvalues: $\begin{vmatrix} 4-\lambda & 8 \\ -2 & -6-\lambda \end{vmatrix} = 0$ implies $\lambda^2 + 2\lambda = 0$, so $\lambda = 0, -2$.

Now, we find the eigenvalues. If $\lambda = 0$,

$$\begin{aligned} 4\alpha_1 - 3\alpha_2 &= 0 \\ 8\alpha_1 - 6\alpha_2 &= 0 \end{aligned}$$

so $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and its multiples are solutions and thus eigenvectors.

If $\lambda = -2$, we obtain $2\alpha_1 - 1\alpha_2 = 0$, and thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector.

Therefore, $\mathbf{x} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$.

Problem 3: *Solve* $\mathbf{x}' = A\mathbf{x}$ *where* A *is* $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$.

Solution: First, we find the eigenvalues: $\begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 2-\lambda & 1 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0$ implies $-(1-\lambda)(2-\lambda)(1+\lambda) = 0$, so $\lambda = 1, 2, -1$.

Now, we find the eigenvalues. If $\lambda = 1$,

$$\begin{aligned} 0\alpha_1 - \alpha_2 + 0\alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ -2\alpha_1 + \alpha_2 - 2\alpha_3 &= 0 \end{aligned}$$

so $\alpha_2 = 0, \alpha_3 = -\alpha_1$, so $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and its multiples are solutions and thus eigenvectors for the eigenvalue $m = 1$.

If $\lambda = -1$, we obtain

$$\begin{aligned} -\alpha_1 - \alpha_2 + 0\alpha_3 &= 0 \\ \alpha_1 + 0\alpha_2 + \alpha_3 &= 0 \\ -2\alpha_1 + \alpha_2 - 3\alpha_3 &= 0 \end{aligned}$$

so $\alpha_2 = \alpha_3 = -\alpha_1$, thus $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is an eigenvector for the eigenvalue $m = -1$.

If $\lambda = 2$, we obtain

$$\begin{aligned} 2\alpha_1 - \alpha_2 + 0\alpha_3 &= 0 \\ \alpha_1 + 3\alpha_2 + \alpha_3 &= 0 \\ -2\alpha_1 + \alpha_2 + 0\alpha_3 &= 0 \end{aligned}$$

so $\alpha_2 = 2\alpha_1, \alpha_3 = -7\alpha_1$, thus $\begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix}$ is an eigenvector for the eigenvalue $m = 2$.

Therefore, $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} e^{-t}$.

Problem 4: *Find the real solutions to the system* $+\mathbf{x}' = A\mathbf{x} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \mathbf{x}$.

Solution: The characteristic equation is $\lambda^2 - 6\lambda + 25 = 0$ so eigenvalues are $\lambda = 3 \pm 4i$.

Complex eigenvectors $\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ satisfy:

$(A - \lambda I)\mathbf{v} = \mathbf{0}$ so $A - \lambda I = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 3+4i & 0 \\ 0 & 3+4i \end{bmatrix} = \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix}$. Multiplying, we have

$$4 \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{0} \to ia_1 + a_2 = 0 \to a_2 = -ia_1$$

Thus, the eigenvector is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$. The complex solution $\mathbf{z}(t) = e^{(3+4i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$ gives real solutions $\mathbf{x}_1 = \mathrm{Re}(z)$, $\mathbf{x}_2 = \mathrm{Im}\, z$.

$$\mathbf{z}(t) = e^{3t}e^{4it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{3t}(\cos 4t + i \sin 4t) \begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{3t} \begin{bmatrix} \cos 4t \\ \sin 4t \end{bmatrix} + ie^{3t} \begin{bmatrix} \sin 4t \\ -\cos 4t \end{bmatrix}$$

$= \mathbf{x}_1 + i\mathbf{x}_2 \to$

$$\mathbf{x}_1 = e^{3t} \begin{bmatrix} \cos 4t \\ \sin 4t \end{bmatrix}$$

$$\mathbf{x}_2 = e^{3t} \begin{bmatrix} \sin 4t \\ -\cos 4t \end{bmatrix}$$

with general solution

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$