The Wronskian

We know that a standard way of testing whether a set of n n-vectors are linearly independent is to see if the $n \times n$ determinant having them as its rows or columns is non-zero. This is also an important method when the n-vectors are solutions to a system; the determinant is given a special name. (Again, we will assume $n = 2$, but the definitions and results generalize to any n.)

Definition 3 *Let $x_1(t)$ and $x_2(t)$ be two 2-vector functions. We define their **Wronskian** to be the determinant*

$$W(\mathbf{x_1}, \mathbf{x_2})(t) \;=\; \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix} \tag{1}$$

whose columns are the two vector functions.

The independence of the two vector functions should be connected with their Wronskian not being zero. For points, the relationship is clear. Using the result mentioned above, we can say

$$W(\mathbf{x_1}, \mathbf{x_2})(t_0) \;=\; \begin{vmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{vmatrix} = 0 \quad \Leftrightarrow \quad \mathbf{x_1}(t_0) \text{ and } \mathbf{x_2}(t_0) \text{ are dependent.} \tag{2}$$

However for vector functions, the relationship is clear-cut *only when x_1 and x_2 are solutions to a well-behaved ODE system (3)*. The theorem is:

We are still considering the system

$$\begin{array}{rl} x' &= a(t)x + b(t)y \\ y' &= c(t)x + d(t)y \end{array} \;, \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3}$$

Theorem 3 Wronskian vanishing theorem.

On an interval I where the entries of $A(t)$ are continuous, let x_1 and x_2 be two solutions to (3) and $W(t)$ their Wronskian (1). Then either

a) $W(t) \equiv 0$ on I, and x_1 and x_2 are linearly dependent on I, or

b) $W(t)$ is never 0 on I, and x_1 and x_2 are linearly independent on I.

Proof. Using (2), there are just two possibilities.

a) $\mathbf{x_1}$ and $\mathbf{x_2}$ are linearly dependent on I; say $\mathbf{x_2} = c_1\mathbf{x_1}$. In this case they are dependent at each point of I, and $W(t) \equiv 0$ on I, by (2).

b) $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent on I, in which case by Theorem 2A they are linearly independent at each point of I, and so $W(t)$ is never zero on I, by (2).   □

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