18.03SC Practice Problems 35

Matrix exponentials

The equation $\dot{\mathbf{u}} = A\mathbf{u}$ (or the matrix A) is
"stable" if all solutions tend to $\mathbf{0}$ as $t \to \infty$.
"unstable" if most solutions grow without bound as $t \to \infty$.
"neutrally stable" otherwise.

A **fundamental matrix** for a square matrix A is a matrix of functions, $\Phi(t)$, whose columns are linearly independent solutions to $\dot{\mathbf{u}} = A\mathbf{u}$.

The fundamental matrix whose value at $t = 0$ is the identity matrix is the **matrix exponential** e^{At}. It can be computed from any fundamental matrix $\Phi(t)$:

$$e^{At} = \Phi(t)\Phi(0)^{-1}.$$

The solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with initial condition $\mathbf{u}(0)$ is given by $e^{At}\mathbf{u}(0)$.

If $\mathbf{q}$ is constant, and A is invertible, then $\mathbf{u_p}(t) = -A^{-1}\mathbf{q}$ is a solution to the inhomogeneous equation $\dot{\mathbf{u}} = A\mathbf{u} + \mathbf{q}$. The general solution is $\mathbf{u_p} + \mathbf{u_h}$, where $\mathbf{u_h}$ is the general solution of the associated homogeneous equation $\dot{\mathbf{u}} = A\mathbf{u}$.

1. In this problem, $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$ and we are interested in the equation $\dot{\mathbf{u}} = A\mathbf{u}$.

(a) Find a fundamental matrix $\Phi(t)$ for A.

(b) Find the exponential matrix e^{At}.

(c) Find the solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(d) Find a solution to $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$. What is the general solution? What is the solution with $\mathbf{u}(0) = \mathbf{0}$?

2. Suppose $\mathbf{u_1}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (the constant trajectory) and $\mathbf{u_2}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$ are solutions to the equation $\dot{\mathbf{u}} = B\mathbf{u}$ for some matrix B.

(a) What is the general solution? What is the solution $\mathbf{u}(t)$ with $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? What is the solution with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

(b) Find a fundamental matrix, and compute the exponential e^{Bt}. What is e^B?

(c) What are the eigenvalues and eigenvectors of B?

(d) What is B?