

## Pset 10 Part II

**Problem 1:** For the  $2 \times 2$  autonomous system

$$\begin{aligned}x' &= x - 2y + \frac{1}{4}x^2 \\y' &= 5x - y - y^2\end{aligned}$$

(a) Find the critical points.

Note: you will get a quartic polynomial; to help you solve it we'll tell you that one root is 0 and another is a positive integer no larger than 5. There are only two critical points, but you'll need to find the other two roots of this polynomial and show they don't give critical points.

(b) Find the linearized system at each critical point. Then carry out the procedure described in the session on Linearization, culminating in one sketch which includes the trajectories near each of the critical points, and a guess at how they all fit together.

**Solution:**

(a) Critical points when  $x' = 0$  and  $y' = 0$

$$y' = 5x - y - y^2 = 0 \Rightarrow x = (y + y^2)/5$$

$$\Rightarrow x' = x - 2y + x^2/4 = (y + y^2)/5 - 2y + (y + y^2)^2/100 = 0$$

$\Rightarrow y^4 + 2y^3 + 21y^2 - 180y = 0$ . By inspection one root is  $y = 0$  and by the hint we find another is  $y = 4$ .

Factoring:  $y(y - 4)(y^2 + 6y + 45) = 0 \Rightarrow y = 0, 4, -3 \pm 6i$ .

Only the real roots give critical points:  $(0, 0)$  and  $(4, 4)$ .

(b) We use the Jacobian.  $\begin{aligned}x' &= f(x, y) \\y' &= g(x, y)\end{aligned} \Rightarrow J(x, y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$

$$\begin{aligned}x' &= x - 2y + x^2/4 \\y' &= 5x - y - y^2\end{aligned} \Rightarrow J(x, y) = \begin{pmatrix} 1 + x/2 & -2 \\ 5 & -1 - 2y \end{pmatrix},$$

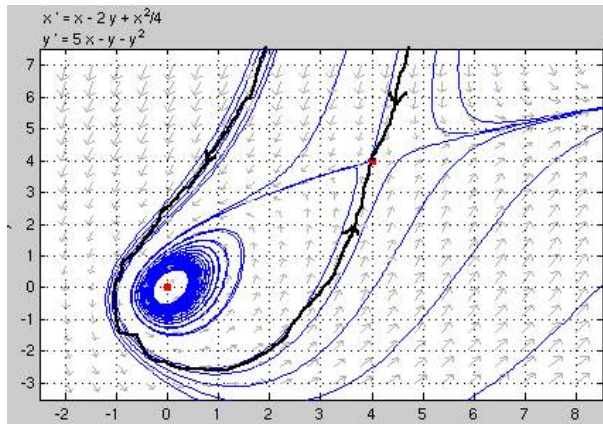
Critical point at  $(0, 0)$ :  $J(0, 0) = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix} \Rightarrow \lambda = \pm 3i$

$\Rightarrow$  linearized system has a **center**  $\Rightarrow$  non-linear system has either a center, spiral source or spiral sink.

Critical point at  $(4, 4)$ :  $J(4, 4) = \begin{pmatrix} 3 & -2 \\ 5 & -9 \end{pmatrix} \Rightarrow \lambda = -3 \pm \sqrt{26}$

$\Rightarrow$  **saddle** (unstable critical point).

(See computer plot below.)



**Problem 2:** *Classify as structurally stable or not stable each of the critical points found in the previous problem. Based on this what can you say for sure about the behavior of the trajectories near each critical point? How does this relate to your hand 'guess-sketch'?*

**Solution:** The linearized critical point at  $(0,0)$  is a center, this is not structurally stable. A linearized center might be (in the nonlinear picture) a center or a spiral in or out.

The linearized critical point as  $(4,4)$  is a saddle. This is structurally stable.

Near the structurally stable critical point our sketch is guaranteed to be qualitatively accurate. Near the not structurally-stable point  $(0,0)$  it is not, although our computer plot gives us some confidence to guess that it is a spiral sink.

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