18.03SC Differential Equations, Fall 2011

Transcript – Lecture 19

Today, and for the next two weeks, we are going to be studying what, for many engineers and a few scientists is the most popular method of solving any differential equation of the kind that they happen to be, and that is to use the popular machine called the Laplace transform. Now, you will get proficient in using it by the end of the two weeks. But, there is always a certain amount of mystery that hangs around it. People scratch their heads and can't figure out where it comes from. And, that bothers them a lot. In the past, I've usually promised to tell you, the students at the end of the two weeks, but I almost never have time. So, I'm going to break that glorious tradition and tell you up front at the beginning, where it comes from, and then talk very fast for the rest of the period.

Okay, a good way of thinking of where the Laplace transform comes from, and a way which I think dispels some of its mystery is by thinking of power series. I think virtually all of you have studied power series except possibly a few students who just had 18.01 here last semester, and probably shouldn't be taking 18.03 anyway, now. But anyway, a power series looks like this: Sum((a)n x^n).

And, you sum that from, let's say, zero to infinity. And, the typical thing you want to do with it is add it up to find out what its sum is. Now, the only way I will depart from tradition, instead of calling the sum some generic name like f(x), in order to identify the sum with the coefficients, a, I'll call it a(x). Now, I want to make just one slight change in that. I want to use computer notation, which doesn't use the subscript (a)n. Instead, this, it thinks of as a function of the discreet variable, n. In other words, it's a function which assigns to n equals zero, one, two, three real numbers. That's what this sequence of coefficients really is. So, the computer notation will look almost the same.

It's just that I will write this in functional notation as a(n) instead of (a)n. But, it still means the real number associated with the positive integer, n, and everything else is the same. See, what I'm thinking of this as doing is taking this discreet function, which gives me the sequence of coefficients of the power series, and associating that with the sum of the power series. Let me give you some very simple examples, two very simple examples, which I think you know. Suppose this is a function one. Now, what do I mean by that? I mean it's the constant function, one. To every positive integer, it assigns the number one. Okay, what's a(x)? What I'm saying is, in other words, in this fancy, mystifying form, is all of these guys are one, what's a of x?

$1 + x + x^2 + x^3$. Look, you are supposed to be born knowing what that adds up to. It adds up to $1/(1 - x)$, except that's the wrong answer. What's wrong about it? It's not true for every value of x. That's only true when x is such that that series converges, and that is only true when $-1 < x < 1$.

So, it's not this function. It's this function with its domain restricted to be less than one in absolute value. What does that converge to? If $x > 1$, the answer is it doesn't converge. There's nothing else you can put here. Okay, let's take another function.

Suppose this is, let's see, 1/n you probably won't know. Let's take one you will know, 1 / n! . Suppose a(n) = 1 / n! , what's a(x)? So, what I'm asking is, what does this add up to when the coefficient here is 1 / n! ? What's Sum[x^n / n!]? It is e^x. And, this doesn't have to be qualified because this is true for all values of x.

So, in other words, from this peculiar point of view, I think of a power as summing the operation, of summing a power series as taking a discreet function defined for positive integers, or nonnegative integers, and doing this funny process. And, out of it comes a continuous function of some sort. And, notice what goes in is the variable, n. But, what comes out is the variable, x. Well, that's perfectly natural.

That's the way a power series is set up. So, the question I ask is, this is a discreet situation, a discreet summation. Suppose I made the summation continuous instead of discreet. So, I want the continuous analog of what I did over there. Okay, what would a continuous analog be? Well, instead of, I'll replace n zero, one, two, that will be replaced by a continued, that's a discreet variable.

I'll replace it by a continuous variable, t, which runs from zero to infinity, and is allowed to take every real value in between instead of being only allowed to take the values of the positive nonnegative integers. Okay, well, if I want to use t instead of n, I clearly cannot sum in the usual way over all real numbers. But, the way the procedure which replaces summation over all real numbers is integration. So, what I'm going to do is replace that sum by the integral from zero to infinity. That's like the sum from zero to infinity of what? Well, of some function, but now n is being replaced by the continuous variable, t. So, this is going to be a function of t.

And, how about the rest of it? The rest I will just copy, x to the n'th. Well, instead of n I have to write t and dt. And, what's the sum? Well, I'll call the sum, what's the sum a function of? I integrate out the t. So, that doesn't appear in the answer. All that appears is this number, x, this parameter, x. For each value of x, like one, two, or 26.3, this integral has a certain value, and I can calculate it. So, this is going to end up as a function of x, just as it did before.

Now, I could leave it in that form, but no mathematician would like to do that, and very few engineers either. The reason is, in general, when you do integration and differentiation, you do not want to have as the base of an exponential something like x. The only convenient thing to have is e, and the reason is because it's only e that people really like to differentiate, e to the something. The only thing is that people really like to differentiate or integrate. So, I'm going to make this look a little better by converting x^t to the base e. I remember how to do that. You write x = e^(ln x) and so x^t will be e^(ln x) * t, if you want.

Now, the only problem is I want to make one more little change. After all, I want to be able to calculate this integral. And, it's clear that if t is going to infinity, if I have a number here, for example, like x equals two, that integral is really quite unlikely to converge. For example, if a(t) were just the constant function, one, the integral certainly wouldn't converge. It would be horrible. That integral only has a chance of converging if x is a number less than one, so that when I take bigger and bigger powers of it, I get smaller and smaller numbers.

Don't forget, this is an improper integral going all the way up to infinity. Those need treatment, delicate handling. All right, so I really want x < 1. Otherwise, that integral is very unlikely to converge. I'd better have it positive, because if I allow it to be negative I'm going to get into trouble with negative powers, see what's minus one, for example, to the one half when t is one half.

That's already imaginary. I don't want that. If you've got an exponential, the base has got to be a positive number. So, I want x to be a positive number. All right, if x in my actual practices going to lie between zero and one in order to make the integral converge, how about log x? Well, log x, if $x < 1$, so $\ln(x) < 0$, and it's going to go all the way down to negative infinity.

So, this means log x is negative. In this interesting range of x, the log x is always going to be negative. And now, I don't like that. The first place I'd like to call this by a new variable since no one uses log x as a variable. And, it would make sense to make it a negative, to make it negative, that is, to write log x is equal to negative s. Let's put it on the other side, in order that since log x is always going to be less than zero, then s will always be positive.

And it's always more convenient to work with positive numbers instead of negative numbers. So, if I make those changes, what happens to the integral? Well, I stress, all these changes are just cosmetic to make things a little easier to work with in terms of symbols. First of all, the a I'm going to change. I don't want to call it a(t) because most people don't call functions a of t. They call them f(t). So, I'll call it f of t. $x = e^{\ln x} = e^{-s}$.

So, x has its name changed to e^{-s}. In other words, I'm using as the new variable not x any longer but s in order that the base be e. t, I now raise this to the t'th power, but by the laws of exponents, that means I simply multiply the exponent by t, and dt. And now, since I'm calling the function f(t), the output ought to be called capital F. But it's now a function, since I've changed the variable, of s. It's no longer a function of x. If you like, you may think of this as a of, what's x? $x = e^{-s}$, I guess. I mean, no one would leave a function in that form. It's simply a function of s. And, what is that? So, what have we got, finally? What we have, dear hearts, is this thing, which I stress is nothing more than the continuous analog of the summation of a power series.

This is the discrete version. This is by these perfectly natural transformations the continuous version of the same thing. It starts with a function defined for positive values of t, and turns it into a function of s. And, this is called the Laplace transform. Now, if I've done my work correctly, you should all be saying, oh, is that all? But, I know you aren't. So, it's okay. You'll get used to it.

The first thing you have to get used to is one thing some people never get used to, which is you put in a function of t, and you get out a function of s. How could that be? You know, for an operator, you put in 3x, and you get out three if it's a differentiation operator. In other words, when you have an operator, the things we've been talking about the last two or three weeks in one form or another, at least the variable doesn't get changed. Well, but for a transform it does, and that's why it's called a transform. So, the difference between a transform and an operator is that for a transform a function of t comes in, but a function of s comes out.

The variable gets changed, whereas for an operator, f(t) goes in and what comes out is g(t), a function using the same variable like differentiation is a typical example of an operator, or the linear differential operators we've been talking about. Well, but this doesn't behave that way. The variable does get changed. That's, in fact, extremely important in the applications.

In the applications, t usually means the time, and s very often, not always, but very often is a variable measuring frequency, for instance. But, so that's a peculiar thing that's hard to get used to. But, a good thing is the fact that it's a linear transform. In other words, it obeys the laws we'd love and like that the Laplace transform-- oh, I never gave you any notation for the laplace transform.

Hey, I'd better do that. Okay, so, some notation: there are two notations that are used. Your book mostly uses the notation that the L(f(t)) = F(s), uses the same letter but with the same capital. Now, as you will see, there are some places you absolutely cannot use that notation. It may seem strange, looks perfectly natural. There are certain laws you cannot express using that notation.

It's baffling. But, if you can't do it this way, you can do it using this notation instead. One or the other will almost always work. So, I'll use my little squiggly notation, but that's what I use. I think it's a little more vivid, and the trouble is that this piles up too many parentheses. And, that's always hard to read. So, I like this better. So, these are two alternate ways of saying the same thing.

The Laplace transform of this function is that one. Okay, well, let's use, for the linearity law, it's definitely best. I really cannot express the linearity law using the second notation, but using the first notation, it's a breeze. The Laplace transform of the sum of two functions is the sum of their Laplace transforms of each of them separately. Or, better yet, you could write it that way. Let's write it this way. That way, it looks more like an operator, L(f) + L(g). And, of the same way, if you take a function and multiply it by a constant and take the laplace transform, you can pull the constant outside. And, of course, why are these true? These are true just because of the form of the transform. If I add up f and g, I simply add up the two corresponding integrals.

In other words, I'm using the fact that the integral, this definite integral, is itself a linear operator. Well, that's the general setting. That's where it comes from, and that's the notation for it. And, now we have to get to work. The first thing to do to get familiar with this is, obviously what we want to do is say, okay, these were the transforms of some simple discreet functions. Okay, suppose I put in some familiar functions, f(t). What do their Laplace transforms look like? So, let's do that. So, one of the boards I should keep stored. Why don't I store on this board? I'll store on this board the formulas as we get them.

So, let's see, what should we aim at, first? Let's first find, and I'll do the calculations on the sideboard, and we'll see how it works out. I'm not very sure. In other words, what's the Laplace transform of the function, one? Well, there's an even easier one. What's the Laplace transform of the function zero? Answer: zero. Very exciting. What's the Laplace transform of one? Well, it doesn't turn out the constant anymore than it turned out to be a constant up there. Let's calculate it. Now, you can do these calculations carefully, dotting all the i's, or pretty carefully, or not carefully at all, i.e. sloppily. I'll let you be sloppy after, generally speaking, you could be sloppy unless the directions tell you to be less sloppy or to be careful, okay?

So, I'll do one carefully. Let's calculate the Laplace transform of one carefully. Okay, in the beginning, you've got nothing to use with the definition. So, I have to calculate the integral from zero to infinity of one, that's the f(t) e^(-st), so I don't have to put in the one, dt. All right, now, let me remind you, this is an improper integral. This is just about the first time in the course we've had an improper integral. But, there are going to be a lot of them over the next couple of weeks, nothing but.

All right, it's an improper integral. That means we have to go back to the definition. If you want to be careful, you have to go back to the definition of improper integral. So, it's the limit, as R goes to infinity, of what you get by integrating only up as far as R. That's a definite integral. That's a nice Riemann integral. So, this is what I have to calculate. And, I have to take the limit as R goes to infinity. Now, how do I calculate that? Well, this integral is equal to, that's easy. It's just integrating. Remember that you're integrating with respect to t. So, s is a parameter. It's like a constant, in other words. So, it's e^(-st), and when I differentiated, the derivative of this would have negative s. So, to get rid of that negative s, so the derivative is e^(-st). You have to put -s in the denominator.

And now, I'll want to evaluate that between zero and R. And, what do I get? Well it is at the upper limit. So, it's e to the minus s times R minus, at the lower limit, it's t is equal to zero, so whatever s is, it's one. And that's divided by this constant up front, negative s. So, the answer is, it is equal to the limit of, as R goes to infinity, of (e^(-sR) - 1) / (-s).

Now, what's that? Well, as R goes to infinity, e^(-2R), or minus 5R goes to zero, and the answer is -1 / -s. So, that's 1/s. And so, that's our answer. Let's put it up here. It's one over s, except it isn't. I made a mistake. Well, not mistake, a little oversight. What's the oversight? This is okay. This is okay. This is okay. This is not okay. This is okay. But that's not okay. What's wrong?

I did slight a verbal hand. Maybe some of you have picked it up and were too embarrassed to correct me, but I said like e^(-2R) obviously goes to zero, and e^(-5R) goes to zero. How about e^(-(-3)R)? Does that go to zero? No, that's e^(3R), which goes to infinity. The only time this goes to zero is if s is a positive number. Minus s looks like a negative number, but it's not, if s is equal to minus two. So, this is only true if s is positive because only if s is positive is this exponent really negative and large, and therefore going to infinity, going to zero as R goes to infinity. So, the answer is not 1/s. It is one over s, s must positive.

Now, once again, here, people don't worry about this sort of thing with power series because it seems very obvious, you know, 1/x, |x| < 1, when it gets to be the Laplace transform, just because the Laplace transform is mysterious, the question is, okay, the Laplace transform is one over s of one, well, Laplace transform of one I understand is one over s if s is positive.

What is it if s is negative? Okay, right down in your little books, this, but that down, what is it if s is negative, and write underneath that, this question is meaningless. It doesn't mean anything. I'll draw you a picture. This is a picture of the Laplace transform of one. It is that. It's one branch of this curve. It does not include the branch on the left. It doesn't because I showed you it doesn't. That's all there is to it. Okay, so I did that carefully. Now I'm going to get a little less careful. What's the L(e^(at))? First of all, in general, the kind of functions for which people like to calculate the Laplace transform, and basically the only ones there will be in the tables are exactly the sort of functions that you used in solving linear equations with constant coefficients.

What kinds of functions entered in there? Exponentials, sines and cosines, but they were really complex exponentials, right? e^t sin(t), but that was really a complex exponential, too, just a little more complicated one, polynomials, and that's about it. t e^t, that was okay, too. These are the functions for which people calculate the Laplace transform, and all the other functions they don't calculate the Laplace transforms. So, I don't mean to disappoint you here. You're going to say, oh, what, that same old stuff? For two more weeks, we've got that same, well, the Laplace transform does a lot of things much better than the methods we've been using.

And, I won't. I'll sell it when I get a chance to, for now, let's just get familiar with it. All right, so while I'm not going to calculate e^(at) for you, because I'd like instead to just prove a simple formula which will just give that, and will also give us e^(at) sin(t). It will give us a lot more, instead. I'm going to calculate a formula for the Laplace transform of this guy if you already know the Laplace transform of it.

Now, see, this falls in that category because this is really e^(at)*1. But, I already know the Laplace transform of one. So that's, if I can get a general formula for this, I'll be able to get the formula for e^(at) as a consequence. So, let's look for this Laplace transform. Now, it's really easy. Let's see, where am I doing calculations? Over here.

Okay, so we've got e. So, I want to calculate the L(e^(at) f(t)). So I'm going to say that's the integral from zero to infinity of e^(at) f(t). And now, the rest I copy. That's the function part of it that goes to the input, and then there's the other part. This part is called the kernel, by the way, but don't worry about that.

However, if you drop it in conversation, people will look at you and say, gee, they know something I don't. And you will. You know that it's the kernel. Okay, well, now, what kind of formula can I be looking for? Clearly, I can only be looking for a formula which expresses it in terms of the L(f(t)). Let's calculate and see what we get. Now, what would you do to that thing to make? Well, obviously, the thing to do is to combine the two exponentials. So, that's going to be the integral from zero to infinity of f(t). e, now, I'd like to put it, to combine the exponentials in such a way that it has, still, that same form, so, I'm going to begin with that negative sign, and then see what the rest of it has to be. What is it going to be?

Well, -st and +at, but I can make that minus a here, and it will come out right. So, it's -st + at, and there are the two parts, those two factors, dt. So, what's that? That's the Laplace transform. If the a weren't there, this would be the Laplace transform of f of t. What is it with the a there? It's the L(f(t)), except that instead of the variable, s --> s - a. I'll give you a second to digest that. Well, you digest it while I'm writing it because that's the answer.

And, the way this is most often used, I have to qualify it for the value. So, if F(s) is good for s positive, the way it would be, for example, if I used the function one here, then to finish that off, then, F(s - a) will be, this will be good when s > a. Why is that? Well, because this is true. This is true. If s - a is positive, that's the condition. That's what this Laplace transform is good. But that simply says that s > a. And, since this doesn't look pretty, let me try to make it look a little bit prettier. So, let's write it.

So, this is assuming F(s), s > 0. Now, this is called something. This is called, well, what would you call it? On the left side, you multiply by an exponential. On the right, you translate. You shift the argument over by a. So, this is called, gulp, the exponential shift. What? Well, I'll call it the formula. The thing before, when we talked about operators, we called it the exponential shift rule or the exponential shift law. But, in fact, this is, in a way, a disguised form of the same law. And, engineers who typically do all their work using the Laplace transform and don't use operators, this is the form of the exponential shift law that they would know.

What you can do with one, you can do with the other. You can now use both. So, what's the answer to e^(at)? Well, the answer is, I'm supposed to, e to the a t times one, the Laplace transform of one is 1/s. And, therefore, what I do is to multiply by e to the a t, I change s --> s - a. And so, that's the answer. Let's see, what else don't we know?

Well, how about sines and cosines? Well, the way to do sines and cosines is by making the observation that this formula also works when a is a complex number. So, can use also for a a complex number, for e^((a + bi)t). The L(e^((a + bi)t)) = 1 / (s - (a + bi)). And again, it will be for s > a. So, let's calculate the Laplace transform of, let's say, well, I've got to cover up something. Okay, so, that's the Laplace transform. I've got to remember that.

So, let's calculate the Laplace transform of, let's say, sin(at) and cos(at). What do you get for that? Well, just for a little variety, we could do it by using that formula, and taking its real and imaginary parts. Since some of you had so much difficulty with the backwards Euler formula, he is a good case where you could use it. Suppose you want to calculate the Laplace transform of cos(at).

Well, I'm going to write that using, I want to calculate using complex exponentials. The way I will do it is by using the backwards Euler formula. So, this is (e^(iat) + e^(-iat)) / 2. Remember, the foreword Euler formula would say e^(iat) = cos(at) + i sin(at). That expresses the complex exponential in terms of sines and cosines. This is the backward formula, which just read it backwards, expressing cosines and sines in terms of complex exponentials instead. Both formulas are useful, almost equally useful, in fact. And anyway, just remind you of it, let's use this one. Okay, what's the Laplace transform, then, of cos(at)?

Well, by linearity, it's equal to one half the Laplace transform of this guy plus the Laplace transform of that guy. And, what are those? Well, the L(e^(iat)) = 1 / (s - ia), and the Laplace transform of the other guy is 1 / (s + ia). Now, of course, this has become out to be a real function. This is real. Every integral is real. This must come out to be real. This looks kind of complex, but it isn't. I know automatically that this is going to be a real function. How I know that? Well, mentally, you can combine the terms and calculate. But, I know even before that. Remember, there are two ways to see that something is real. You can calculate it and see that its imaginary part is zero, hack, or without any calculation, if you change i --> -i, and you get the same thing, it must be real.

Now, if I change i to minus i in this expression, what happens? If I change i to minus i, this term turns into that one, and this one turns into that one. Conclusion: the sum of the two is unchanged. And therefore, this is real. Well, of course, in the time I took to make that argument, I could have actually calculated it. So, what the heck, let's calculate it? So, you do the high school thing, and it's this guy plus that guy on top, which makes 2s. I on the bottom is the product of those, which by now you should know the product of two complex numbers. A product of a number and its complex conjugate is the sum of the squares.

So, what's the answer? The twos cancel, and the answer is that the L(cos(at)) = s / (s^2 + a^2). And, that will be true as, in general, it's true up there for positive values of s only. And, the sin(at) , you can calculate that in recitation tomorrow. The answer to that is a / (s^2 + a^2). You would get the same answers if you took the real and imaginary parts of that expression.

It's another way of getting at the recitations tomorrow; we'll get practice in calculating other functions related to these by using these formulas, and also from scratch directly from the definition of the Laplace transform. Well, there are two things which we still should do. The first is I want to get you started with calculating inverse Laplace transforms. And, the reason for doing that is, in other words, I've started with f(t), and we've been focusing on what is capital F(s)? But, you will find that when you go to solve differential equations, by far, the hardest part of the procedure is you get F of s. The Laplace transform of the answer, and you have to convert that back into the answer in terms of t that you were looking for. In other words, the main step in the procedure that you are going to be using for solving differential equations is, and the hardest part of the step will be to calculate inverse laplace transforms.

Now, you think that could be done by tables, but, in fact, it can't unless the tables are too long to be useful. You have to do a certain amount of work yourself. And, the certain amount of work that you have to do yourself involves partial fractions decompositions. And, in case you were wondering which you are not, the reason you learned partial fractions in 18.01 was not to learn those silly integrals, but he learned it so that when you got to 18.03 you would be able to calculate, solve differential equations by using Laplace transforms. Sorry. That's life. Now, so a certain amount of the recitation time tomorrow will be devoted to reminding you how to do partial fractions since you haven't done it in a while, and I assume, yeah, we had that, I think.

Okay, now, they also remind you of the most efficient method, which about half of you have had, and the rest think you might have had, but really aren't sure. So, here's the answer. We want to find out what it's inverse Laplace transform is. What you have to do, it normally won't be in the tables like this. You have to put it in a form in which it will be in the tables. As you do that, you have to make partial fractions decompositions, which, to do it quickly, so if you don't know what I'm doing now, or you think you once knew but don't quite remember, go to recitation tomorrow. To get the coefficient here, I cover up s, and I put s = 0 because that's the law.

To get this coefficient, I cover up s plus three and I put s equals a negative three because that's what you're supposed to do. Put s equal negative three, you get minus one third. This is equal to that. In this form, I don't know what the inverse Laplace form is, but in this form, I certainly do know with the inverse Laplace transform because the inverse Laplace transform is linear, and because each of these guys especially occurs in those tables. Well, what's this?

Well, it's whatever the Laplace transform of, inverse Laplace transform of one over s is multiplied by one third. Well, the inverse Laplace transform of one over s is one. So, it's one third times one. How about the other guy? Minus one third, the inverse Laplace transform of one over s plus three, that's this formula. a is negative three, and that makes e^{-3t}. So, if this was the Laplace transform of the solution to the differential equation, then the solution in terms of t was this function. Now, you'll get lots of practice in that. All I'm doing now is signaling that that's the most important and difficult step of the procedure, and that, please, start getting practice.

Get up to snuff doing that procedure. Okay, in the time remaining, I want to add one formula to this list, and that is going to be the Laplace transform of, we still haven't done polynomials. And now, to polynomials, because the Laplace transform is linear, all I have to do is know what the Laplace transform of, the individual term of a polynomial. In other words, what the $L(t^n)$, where n is some positive integer? Well, let's bravely start trying to calculate it. Integral from 0 to infinity of $[t^n e^{-st}\, dt]$. Now, I think you can see that the method you should use is integration by part because this is a product of two things, one of which you would like to differentiate a lot of times, in fact, and the other won't hurt to integrate it because it's very easy to integrate. .

So, this factor is going to be the one that's to be differentiated, and this is the factor that will be pleased to integrate it. Let's get started and see what we can get out of it. Well, this time I'm going to be, well, I'd better be a little careful because there's a point here that's tricky. Okay, the first step of integration by parts is you only do the integration. You don't do the differentiation. Remember, the variable is t. The s is just a parameter. It's just a constant. It's hanging around, not knowing what to do. Okay, so the first step is you don't do the differentiation. You only do the integration. Evaluate it between limits, and then you put a minus sign before you forget to do it.

And then, integral zero to infinity. Now you do both operations. So, it's $n\, t^{n-1}$, and you also do the integration. Okay, let's consider each of these pieces in turn. Now, this piece, well, there's no problem with the lower limit, zero, because when t is equal to zero, this factor is zero, and the thing disappears as long as n is one or higher. So, it's minus zero here at the lower limit. The question is, what is at the upper limit? So, what I have to do is find out, what is the limit? The limit, as t goes to infinity, that's what's happening up there, of $t^n e^{-st} / (-s)$.

Well, as t goes to infinity, this goes to infinity, of course. This had better go to zero unless I want an answer, infinity, which won't do me any good. If this goes to zero, s had better be positive. So, I'd better be restricting myself to that case. Okay, so let's assume that s is positive so that this minus s really is a negative number. Okay, then I have a chance. So, this is going to be the limit. Let's write it in a more familiar form with that down below. So, it's t^n. That's going to infinity. But, the bottom is e^{-st}. But now, it's $+st$. And, that's going to infinity, too, because s is positive. So, the two guys are racing, and the question is, oh, I lost a minus s here. So, oh, [GRUNTS] equals $-1 / s$. How's that?

So, the question is only, which guy wins? In the race to infinity, which one wins, and how do you decide? And, the answer, of course, is that's the bottom that wins. The exponential always wins, and it's because of L'Hopital's rule. You differentiate top and bottom. Nothing much happens to the bottom. It gets another factor of s, but the top goes down to $t^{n - 1}$. L'Hopital it again, and again, and again, and again, and again until finally you've reduced the top to t to the zero where it's defenseless and just sitting there, and nothing's happened to the bottom.

It's still got e^(st), and that goes to infinity. So, the answer is, this is zero by n applications of L'Hopital's rule. Or, if you're very clever, you can do it in one, but I won't tell you how. So, the answer is that this is zero. At the upper limit, it's also zero at least if s is positive, which is the case we're considering. That leaves the rest of this. All right, let's pull the constants out front. That's plus. Two negatives make a plus. n / s, now, what's left? Integral from 0 to infinity of [t^(n - 1) e^(-st) dt]. But, what on Earth is that? That is n/s times the Laplace transform of t^(n-1). We got a reduction for it. We don't get the answer in one step.

But, we get a reduction formula. And, it says that the Laplace transform, let me write it this way for once. The first way is now better, is equal to n over s times the Laplace transform of n minus t to the n / s * L(n - t^(n - 1)). Okay, the next step, this would be n / s * (n - 1) / s * L(t^(n - 2)).

If I can continue, I finally get in the top n times n minus one times all the way down to one divided by the same number of s's, n of them, times the Laplace transform of t to the zero, finally. See, one, zero, n minus one. And so, what's the final answer? It is n factorial over s to the what power? Well, the Laplace transform of this is 1/s. So, the answer is it's s^(n + 1), n of them here plus an extra one coming from the one over s here. And, that's the answer. The L(t^n), oddly enough, is more complicated, and looks a little different from these. It's n! / (s^(n + 1)). And, with that, you can now calculate the Laplace transform of anything in sight, and tomorrow you will.