18.03SC Differential Equations, Fall 2011
Transcript – Convolution and Green's Formula

PROFESSOR: Hi, everyone. Welcome back. In this problem, I'd like to take a look at convolution and Green's formula. Specifically, for part one, we're just asked to compute the convolution of t with itself, t. In the second part, we're asked to compute the convolution of e to the minus kt with another exponential, e to the at. And then in part B, we're to use the result from part A as well as Green's formula to solve x dot plus kx equals e to the at. And specifically, we're interested in the rest initial condition solution to this formula. OK. So I'll let you think about this problem, and I'll be back in a minute.

Hi, everyone. Welcome back. OK, so for part one, we're asked to compute the convolution of two functions, t with itself-- and I'll just quickly write down the formula for convolution. So if I have two functions, f and g, then the convolution is defined as, in this case, the integral from zero to t of f, so the function f evaluated at, we're going to use tau as the integration variable, multiply by g, t minus tau, d tau. So notice how the variable t appears in two places in this formula. It appears in one place in the argument of g, and it appears in the bound of integration. Meanwhile, tau is the variable we're integrating over, and tau appears in two places as well, one in f and one in g.

So if we're interested in t convolved with t, we have the integral from zero to t of tau times t minus tau. So notice how f is just t and g is t, so when we insert tau and t minus tau into the function, we end up just getting tau times t minus tau, d tau. And just expanding this out, we have t times tau minus tau squared, d tau. So in the integral, the variable t always appears just as a constant. So in this case, we have t, tau integrated becomes t squared over 2. When I integrate from zero to t, tau squared integrates to 1/3 t cubed, and zero drops out from the other end of the integral. So we end up with t cubed, 1/2 minus 1/3, which is equal to 1/6 t cubed. So there's the answer we're looking for.

For part B, or for question two, again we have another computation to do. So we have more integrals to work out. And in this case, it's two exponentials. So we have zero to t. The first one is going to be e to the k tau. And the second one is e to the at minus tau, d tau. So we can just expand things out, and I get e to the at times e to the minus a plus k tau d tau, zero to t. And again, because t is just a dummy variable in this integral, we're not integrating over t, I can just think of e to the at as a constant. So I'm really just integrating the function e to the minus a plus k tau. And when I integrate this function, I get 1 over minus a plus k, e to the minus a plus k, tau evaluated at the bounds, zero and t. So the negative sign here just flips the bounds. So we have a plus k. Substituting in zero just gives us 1, and then substituting in t gives us e to the minus a plus kt. And I can expand out, multiply out, e to the at. And when the dust settles, we have e to the at minus e to the minus kt.

So this is just the computation of the convolution e to the minus kt and e to at. So note now if, for example, k were to equal negative a, we would have zero in the denominator and zero on the numerator. So one way to compute the special k case when k is equal to negative a is to use L'Hopital's Rule and differentiate the top and the bottom. For example, with respect to k. Another alternative way

of computing the special case when a is equal to negative k is just to plug in e to the k, and then just work out this integral, and you'll obtain a different answer.

And now lastly, for part B, we're interested in the differential equation x dot plus kx equals e to the at. And we're interested in finding the particular solution that has rest initial conditions. So for example, x of zero is equal to zero. And we want to use Green's formula. So just recall that there's the impulse response formula for the weight function w, which solves the same differential equation, but with the delta function on the right hand side. And w, in this case, as we've seen in lecture, is actually e to the negative kt when t is bigger than zero, and at zero when t is less than zero.

So notice how e to the minus kt is exactly the function that we convolved in part A. Specifically, we convolved it with e to the at. So for example, Green's formula says that the particular solution that has rest initial conditions is going to be the wave function convolved with the right hand side of the differential equation, e to at. In this case, on the domain of integration for the convolution, w is just e to the minus kt. So we have e to the minus kt convolved with e to the at.

So Green's formula gives us the solution to this differential equation, which has rest initial conditions. And specifically-- we've already worked this out from part A, so I can just write down what the answer is. It's 1 over a plus k, e to the at, minus e to the minus kt. And just as a quick check to make sure that we've done everything correctly, we can plug in t equals zero. And when t equals zero, we have 1 minus 1, which gives us zero. So indeed, this x is the solution to this differential equation, which has rest initial conditions.

I'd just like to conclude this problem. Quick recap. We worked out several convolutions, and specifically we were able to use Green's formula in addition to the convolution that we worked out to compute the solution with rest initial conditions to an ODE. Moreover, Green's function is very useful because notice how we could have computed the convolution for any right hand side function here. So in some sense, we're able to generalize and find solutions to differential equations which have arbitrary right hand side forcings, not just sines and cosines, periodic functions, or other simple functions which we've been looking at in the past.

So I'd like to conclude here, and I'll see you next time.

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