Definition and Properties

1. Definition

The **convolution** of two functions f and g is a third function which we denote $f * g$. It is defined as the following integral

$$(f * g)(t) = \int_{0^-}^{t^+} f(\tau)g(t-\tau)\,d\tau \quad \text{for } t > 0. \tag{1}$$

We will leave this unmotivated until the next note, and for now just learn how to work with it.

There are a few things to point out about the formula.

- The variable of integration is τ. We can't use t because that is already used in the limits and in the integrand. We can choose any symbol we want for the variable of integration –it is just a *dummy* variable.
- The limits of integration are 0^- and t^+. This is important, particularly when we work with delta functions. If f and g are continuous or have at worst jump discontinuities then we can use 0 and t for the limits. You will often see convolution written like this:

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

- We are considering **one-sided convolution**. There is also a two-sided convolution where the limits of integration are $\pm\infty$.
- (**Important.**) One-sided convolution is only concerned with functions on the interval $(0^-, \infty)$. When using convolution we never look at $t < 0$.

2. Examples

Example 1 below calculates two useful convolutions from the definition (1). As you can see, the form of $f * g$ is not very predictable from the form of f and g.

Example 1. Show that

$$e^{at} * e^{bt} = \frac{e^{at} - e^{bt}}{a-b}, \quad a \neq b; \qquad e^{at} * e^{at} = t\,e^{at}$$

Solution. We show the first; the second calculation is similar. If $a \neq b$,

$$e^{at} * e^{bt} = \int_0^t e^{a\tau} e^{b(t-\tau)}\, d\tau = e^{bt} \int_0^t e^{(a-b)\tau}\, d\tau = e^{bt} \left. \frac{e^{(a-b)\tau}}{a-b} \right]_0^t = e^{bt} \frac{e^{(a-b)t} - 1}{a-b} = \frac{e^{at} - e^{bt}}{a-b}.$$

Note that because the functions are continuous we could safely integrate just from 0 to t instead of having to specify precisely 0^- to t^+.

The convolution gives us a formula for a particular solution y_p to an inhomogeneous linear ODE. The next example illustrates this for a first order equation.

Example 2. Express as a convolution the solution to the first order constant-coefficient linear IVP.

$$\dot{y} + ky = q(t); \quad y(0) = 0. \tag{2}$$

Solution. The integrating factor is e^{kt}; multiplying both sides by it gives

$$(y\, e^{kt})' \;=\; q(t)e^{kt}.$$

Integrate both sides from 0 to t, and apply the Fundamental Theorem of Calculus to the left side; since we have $y(0) = 0$, the solution we seek satisfies

$$y_p\, e^{kt} \;=\; \int_0^t q(\tau)e^{k\tau}\, d\tau; \qquad (\tau \text{ is the dummy variable of integration.})$$

Moving the e^{kt} to the right side and placing it under the integral sign gives

$$\begin{aligned} y_p \;&=\; \int_0^t q(\tau)e^{-k(t-\tau)}\, d\tau \\ y_p \;&=\; q(t) * e^{-kt}. \end{aligned}$$

Now we observe that the solution is the convolution of the input $q(t)$ with e^{-kt}, which is the solution to the corresponding homogeneous DE $\dot{y} + ky =$, but with IC $y(0) = 1$. This is the simplest case of **Green's formula**, which is the analogous result for higher order linear ODE's, as we will see shortly.

3. Properties

1. **Linearity:** Convolution is *linear*. That is, for functions f_1, f_2, g and constants c_1, c_2 we have

$$(c_1 f_1 + c_2 f_2) * g = c_1(f_1 * g) + c_2(f_2 * g).$$

This follows from the exact same property for integration. This might also be called the **distributive law**.

2. **Commutivity:** $f * g = g * f$.
Proof: This follows from the change of variable $v = t - \tau$.
Limits: $\tau = 0^- \Rightarrow t - \tau = t^+$ and $\tau = t^+ \Rightarrow t - \tau = 0^-$
Integral: $(f * g)(t) = \int_{0^-}^{t^+} f(\tau) g(t-\tau)\, d\tau = \int_{0^-}^{t^+} f(t-v) g(v)\, dv = (g * f)(t)$

3. **Associativity:** $f * (g * h) = (f * g) * h$. The proof just amounts to changing the order of integration in a double integral (left as an exercise).

4. Delta Functions

We have

$$(\delta * f)(t) = f(t) \quad \text{and} \quad (\delta(t-a) * f)(t) = f(t-a). \tag{3}$$

The notation for the second equation is ugly, but its meaning is clear.

We prove these formulas by direct computation. First, remember the rules of integration with delta functions: for $b > 0$

$$\int_{0^-}^{b} \delta(\tau) f(\tau)\, d\tau = f(0).$$

The formulas follow easily for $t \geq 0$

$$\begin{aligned}
(\delta * f)(t) &= \int_{0^-}^{t^+} \delta(\tau) * f(t-\tau)\, d\tau = f(t-0) = f(t) \\
(\delta(t-a) * f)(t) &= \int_{0^-}^{t^+} \delta(\tau - a) * f(t-\tau)\, d\tau = f(t-a).
\end{aligned}$$

5. Convolution is a Type of Multiplication

You should think of convolution as a type of multiplication of functions. In fact, it is often referred to as the *convolution product*. In fact, it has the properties we associate with multiplication:

- It is commutative.

- It is associative.
- It is distributive over addition.
- It has a multiplicative identity. For ordinary multiplication, 1 is the multiplicative identity. Formula (3) shows that $\delta(t)$ is the multiplicative identity for the convolution product.