18.03SC Differential Equations, Fall 2011
Transcript Pure Resonance

PROFESSOR: Welcome to this recitation on pure resonance. So here we're given an operator pd equals d squared plus 4i, where d is the differential operator and i is the identity operator. And you're asked to consider the equation pd applied to x equals f0 cos omega t, where f0 is a constant. So the first question is what is the natural frequency of the system.

The second one is to use the exponential response formula to solve for pd x equals f0 cos omega t. And here you need to be careful and do it for both cases omega equal to 2 and omega equals not equal to 2.

And the last question is just to sketch the graph for the response of this system pd x equals cos 2t, with the initial conditions x of 0 and x dot of 0 equals to 0, basically, rest initial conditions. So why don't you pause the video, take a few minutes, and work through this problem.

Welcome back. So first what is the natural frequency of this system? So let's just rewrite our system here. This is the left-hand side. So basically, this just gives us an x dot dot plus 4x on the left-hand side. So the system that we're solving is simply x dot dot plus 4x equals f0 cos omega t.

So the first question asks us for the natural frequency of this system. The natural frequency of this system can be found regardless of what you have the right-hand side, just by looking at the characteristic polynomial of your equation. The characteristic polynomial here would be s squared plus 4. When this characteristic polynomial is equal to 0, we can solve for s and find what are the natural frequencies of the system, if basically we get complex solutions, which is the case here.

Gives us a square equal minus 4. So s equals plus or minus i2. So the natural frequency of our system would be omega equals 2, because we only consider frequencies that are positive here.

Second part. Now, we're asked to look at the full system with the forcing on the right-hand side. And using the exponential response formula, find one solution to this system. So here we're talking about a particular solution with the exponential response formula.

So what does the ERF tell us? The ERF, if you recall here, the base of it for this system for example, is the fact that cosine is the real part of the exponential i omega t. So we can rewrite this whole equation as x dot dot plus 4x equals f0 exponential i omega t.

And we would get then a particular solution, if I ignore any particular value of omega at this point, which would have the form of the amplitude that we have on the right-hand side at 0, exponential omega t, which is basically our forcing, over the characteristic polynomial of the equation. So s squared plus 4, evaluated at the frequence here that would appear at the forcing in the exponential form, so with the i omega t.

So here you can see right away that we would have a problem. If you were using this formula, if i omega t was a pole or basically a 0, to this characteristic polynomial. And so that's why you were asked to be careful with the value of omega equals to 2 or not equal to 2.

So here let's consider omega not equal to 2, so that I can actually write down 1 over p i omega, because we know that p i2 is equal to 0. So if omega is not equal to 2, were out of the danger zone. And from this point, we can just basically plug in our values, i omega t, and p i omega would just give us 4 minus omega squared.

So here again, that omega equals 2 danger zone approaches, where we would be dividing by 0 if we didn't take the constraint omega not equal to 0. So this is the complex form of this particular solution. But we're dealing with a real value problem, so we want to take the real part of this to have the solution to the problem we were given. And so that would just give us f0 4 minus omega squared cosine omega t.

So now let's take the case omega equals to 2. OK So what happens? If omega equals to 2, this formula that you're given fails, and you need to seek for the derivative of the characteristic polynomial. And we basically have 2i equals to 2. So what about p prime of 2i? So p prime of s is simply 2s. So if we evaluate p prime at 2i, we simply have 4i, which is not equal to 0.

So at this point we can use the resonant exponential response formula that you saw. Just change my chalk. We're here. We would again, same trick, the cosine is just the real part of the exponential. So we can use this formula. And we have now to introduce a t f0 exponential i omega t, because we're solving here for the complex value equation. And now we can divide by the p prime evaluated at 2i, which is 4i. And so basically, I can end up with a minus i at the numerator.

So to take now the real value solution, we need again to take the real part of zp. So here now we have an i, so we need to be careful. We're going to have solution in sine. So let me just write down what know. t f0 over 4. This with the Euler formula would be cosine plus i sine. The i sine would be multiplying this i, the 2 minus would cancel out. And so we would end up with sine omega t, t f0 over 4. And this would then give us the solution.

And here, note that I actually chose the value omega equals to 2, so we can even be more explicit. For this case, we actually have omega equals to 2t.

So the last part of the problem was to sketch the solution for the initial conditions, x of 0 or x dot of 0 equal to 0, so the rest initial conditions. So here are two ways to proceed. The long way would be to seek the solution to the homogeneous equation without the right-hand side, the forcing cosine, introduced two constants of integration, and then seek these constants of integration on the general solution. And you would find that these two constants of integration would be 0 with these initial conditions.

The other fast way to test your particular solution and verify that it actually does satisfy the initial conditions that you were given, and so you can then write away the solution as being simply sine 2t. Here you can see that at 0, we would have basically a 0. And then if you do a differentiation, you just need to be careful here, because you have a product function, and you end up also with a 0.

So this actually is our general solution for this particular initial condition, And to sketch this, we can grow. So here if I just pick f0 equal to 1, I'm just going to do t/4 for the envelopes. At t equals 0, we start with 0. And we know that we're going to have the first extrema at p/4 and the first 0 at pi/2.

And so basically, we end up with something like that. So basically, it's sine of circular frequency 2, and with an envelope prescribed by t/4. Or if we had another value at 0, it would be f0 t/4. So the oscillation is ongoing as t goes to infinity with an envelope that diverges to infinity. So this is basically a solution that would not be convergent to 0.

So this ends this recitation. And before I finish, I just want to point out that the fact that it diverges is due to the fact that we are forcing this system very close to its natural frequency. And so this is a typical phenomenon that you can associate with a resonance, because we're basically forcing a system close to its natural frequency. So it's having this huge amplification in the response. And that's what these increasing envelopes mean.

So this ends this recitation. And the key here was to realize how to use your exponential response formula, how to move on to use the resonance exponential response formula by testing for the first order derivative. And if that test failed, you would be going to higher orders. And then, given an initial condition, how to basically sketch the function and have a physical understanding of what the resonance means. That ends this recitation.