18.03SC Differential Equations, Fall 2011
Transcript – Amplitude and Phase: Second Order II Applet Applet

PROFESSOR: Welcome to the applet, Amplitude and Phase Second Order II. This applet is a sibling of the applet, Amplitude and Phase First Order, which I used to introduce the MIT Mathlet collection. It's color coding, placement conventions, and it's functionalities are identical to that one.

This applet represents a mechanical system driven by the motion of the far end of the dashpot. Let's begin by animating the system to see how this works. You can see the dashpot is moving up and down sinusoidally, driving a mass in yellow. And the mass is also constrained by a spring at the top, attached to a fixed wall at the top. Perhaps we should animate this again, so you can see the whole thing at work.

This applet, and its siblings, show only the steady state or periodic solutions to these equations. They don't allow you to pick initial conditions. They don't represent the effect of transients.

In the applet, x represents the position of the mass, and you can see it read off on this scale. It's represented in yellow on the graphing window here. And we declare x to be zero when the spring is exerting no force on the mass. It's at rest. We declare x to be the output signal of the system in the mathlet.

Y denotes the position of the plunger, and it's read off by this scale here. We declare y to be the input signal of the system.

Now the force exerted by a dashpot is proportional to the relative velocity of the cylinder and the piston sliding inside of it. The proportionality constant is called b. As a consequence, the right-hand side of the equation controlling the position of the mass is b,y dot -- b times the time derivative of y.

This is a good example in which the right-hand side of a linear equation in standard form is not just the input signal. In this case, it's not even a multiple of the input signal. It's a multiple of the derivative of the input signal.

The most important case to study is that in which the input signal is sinusoidal, and that's what is represented here. In the applet, we take the amplitude of the input signal to be one. If the amplitude of the input signal is doubled, so is its derivative, and so by superposition the amplitude of the output signal would also be doubled. So setting the input's amplitude equal to one isn't really a restriction.

In this situation, when the input amplitude is one, the output amplitude is the same as the gain of the system. We can see how this output amplitude depends upon the input frequency by opening the Bode Plot window here. This opens two windows.

The top one represents the amplitude of the system response, the gain, as a function of omega. And we can start with omega small. When omega equals zero, the plunger isn't moving at all, and the mass has no reason to move either. So x equals zero, the spring is relaxed, there's no motion.

When omega increases, the system response becomes greater. But you'll notice something interesting. In this system, when omega is small, the system response leads the input signal. That is to say, the phase lag is negative. And that's represented here.

This is a little confusing. This is a graph of the negative of the phase lag or the phase gain. And that's a positive angle, in this system, for omega small. You can see the effect. The mass seems to be pulling the plunger, although that's not actually what's happening.

When omega increases, the amplitude of the system response increases until a critical moment when omega takes on some critical value. This is the resonant frequency of the system. And at this point, the system response is identical to the input signal. Shall we see what this looks like by animating the system? Now the plunger seems to be locked to the piston. That's not actually the case, but the system is simply operating in harmony here. It's the resonant frequency.

When omega increases still further, then the system response falls off. The phase lag becomes positive, the response falls behind the input signal. And that's all visible from the Bode plot pictures.

As I look at this, I notice something interesting. When the system response reaches a maximum or a minimum, that's exactly the same moment as when the system response curve crosses the input signal curve. So in other words, the amplitude of the output signal equals the value of the input signal at the moment when they cross. Let's see if that is always the case when I change the frequency here. That always seems to be the case. Very interesting.

And in fact, perhaps we should change the values of b and k to see whether that continues to be the case. Yes it just seems to drag up and down there. And similarly, if I change the value of the spring constant, k, again that peak seems to move up and down along the blue curve. Very odd. We'll come back to that.

Here are some further observations that you can see from experimenting with this applet. First of all, let's watch what happens if I change the value of the damping constant, b. I'm going to watch the Bode plot over here, the amplitude Bode plot.

Well, it changes, but one thing that doesn't change is the position of the resonant peak there. In other words, the resonant frequency seems to be independent of the value of b, of the damping constant.

The second thing you can observe as I vary b is when b is large, the resonant hump is quite broad, but as b becomes smaller, that spike becomes narrower and narrower and narrower.

And the third thing you can notice from looking at this is as b gets to be small, the flip between phase lag of close to minus pi over 2 to a phase lag of close to plus pi over 2 happens more and more abruptly as I change the frequency from something small and cross through that resonant peak. Now the system response is ahead of the input signal. But it very quickly flips to being behind it as you cross the resonant frequency. And that transition happens more and more rapidly as b gets to be small.

Each one of these three observations can be verified by calculation.

The relationship between the gain and the phase lag can be illustrated very well using the Nyquist plot, which I'll open using this key. This is a plot of the complex plane, and on it is drawn a complex number in yellow. And that complex number has a magnitude and an angle. The magnitude is the gain. This yellow strut is the same length as this radius vector. And the angle is the negative of the phase lag. It's the phase gain.

So when the angle's positive here, it goes through zero here and becomes negative down here, when the phase lag becomes positive. So you can see why it is that the resonant peak occurs at exactly the same instant as when the phase lag is zero. That says that this trajectory, this Nyquist plot, goes through this point, which is the point 1 in the complex plane, where the angle is zero and the magnitude is 1.

This trajectory is in fact independent of either or both of the system parameters. If I change them, many things change, but the shape of this trajectory, the Nyquist plot, is independent of those two system parameters. And in fact, it's a circle of radius 1/2 and center 1/2. You can verify that by calculation as well.

That fact explains the observation we made earlier, that the amplitude of the system response, the gain, seems to be equal to the value of the input signal when that maximum is achieved. In other words, the claim is that the gain equals the cosine of the phase lag. The gain is the cosine of the phase lag.

This curve is given by the equation, the radius equals the cosine of the angle. r equals cosine theta is a polar equation for this particular circle. And so the fact that this is a circle is the same as the fact that we observed earlier, that the solution curve crosses the input curve just when it reaches a maximum or a minimum.

Well, in addition to the spring system shown in this applet, this very same equation models the response of an A/M radio receiver. The input signal is the incoming radio signal, which contains electromagnetic waves of many different frequencies. The output signal feeds to the amplifier. You tune the radio to a particular angular frequency by adjusting k, the spring constant, which in the circle model is the capacitance, so that the resonant frequency is the desired frequency that you want to tune to.

Then the output generated by the other frequencies is much less than the output generated by frequency omega r. How much less depends on how sharp the resonance spike is. The sharper the spike, the better the tuning. You can make the spike sharper by decreasing the value of b, which in the A/M radio model, is the resistance.