Exponential Input

Let's consider the case where the driving function is an *exponential* Ae^{at}, where A and a are constants. We will allow A and a to be complex, so this will also be useful for dealing with *sinusoidal* driving functions, e.g., $F_{ext}(t) = 3\cos(2t)$.

Let's try to solve a particular example.

Example. Find the general solution to

$$x'' + 8x' + 7x = 9e^{2t}.$$

We have no method yet, but we can at least try to guess (the method of optimism). We hope that we can get a solution which is similar in form to the right hand side. So let's guess

$$x(t) = Ae^{2t},$$

where A is an unknown constant. Substituting we get

$$x'' + 8x' + 7x = 4Ae^{2t} + 16Ae^{2t} + 7Ae^{2t} = 27Ae^{2t}.$$

Success! Setting $A = 1/3$, we have a solution $x_p = \frac{1}{3}e^{2t}$.

We are not done yet, since we want the general solution. Now we only need to solve the homogeneous equation, and then we can apply Superposition II. The associated homogeneous equation is

$$x'' + 8x' + 7x = 0.$$

The characteristic polynomial is

$$p(r) = r^2 + 8r + 7 = (r+7)(r+1).$$

The roots are -7 and -1, so we deduce that

$$x_h = c_1 e^{-7t} + c_2 e^{-t}$$

is the general solution to the homogeneous equation. Thus the general solution to the original equation is

$$x = x_h + x_p = c_1 e^{-7t} + c_2 e^{-t} + \frac{1}{3}e^{2t}.$$

Make sure you understand why the first two terms have a parameter and why the third does not.

We can try this same approach to the general form

$$mx'' + bx' + kx = Be^{at},$$

where B and a are constants. Again, we use the method of optimism, and try a solution of the form $x(t) = Ae^{at}$, A being an unknown constant. Substituting, we find

$$\begin{aligned} mx'' + bx' + kx &= m \cdot a^2 Ae^{et} + b \cdot aAe^{at} + k \cdot Ae^{at} \\ &= (ma^2 + ba + k)Ae^{et}. \end{aligned}$$

Thus to be a solution, we must set

$$A = \frac{B}{ma^2 + ba + k}.$$

Notice that the denominator in this expression can be written succinctly as just $p(a)$, where p is the characteristic polynomial we saw in the context of the homogeneous equation. We have

Exponential Response Formula (ERF). Consider the second order equation

$$mx'' + kx' + bx = Be^{at},$$

and let $p(r) = mr^2 + kr + br$ be its characteristic polynomial. Then

$$x(t) = \frac{B}{p(a)} e^{at}$$

is a particular solution, as long as $p(a) \neq 0$.

(You might worry about the restriction $p(a) \neq 0$—and you should. We'll come back to that shortly.) This formula works essentially unchanged for higher order equations too—we'll see that in a future session.

Don't forget that this only gives a single particular solution. For the general solution, you must still solve the associated homogeneous problem and then apply Superposition II.