18.03SC Differential Equations, Fall 2011
Transcript – Damped Vibrations Applet

The damped vibrations applet illustrates several different concepts related to differential equations. Initial conditions for second order equations, the phase plane for autonomous equations, and damping conditions for second order homogeneous linear equations.

The differential we're studying appears at upper right. And the parameter names reflect the fact that this equation describes a mass spring dashpot system, with mass m, dashpot constant b, and spring constant k.

The right hand side of the equation is zero. There's no forcing term. The equation is homogeneous.

The sliders at bottom control the values of the system parameters. With the selected values for m, b, and k, solutions are damped vibrations. I can let the solution evolve by grabbing the time slider or by using the animation key, which also serves to stop the animation.

If the value of the solution gets too small, I can blow the picture up using the zoom slider at the upper left. Moving it to the right squeezes the solutions. Moving it to the left makes them larger.

The window on the left lets you set the initial condition of the system. There are two components to the initial condition of a second order equation. Position and velocity, that is x and x dot. Together they form the coordinates of a point on this plane. I can grab the point and change both x and x prime simultaneously.

The x-coordinate is written vertically, because that's how it's written on the right hand graphing window. And you can see that they keep pace with each other. The horizontal component is x dot, the initial velocity. That's positive when we're to the right, and becomes negative when we're to the left.

The left window represents phase space. For any time t, the values of x and x dot are the coordinates of a point in phase space. And as the solution evolves through time, that point follows a path, sweeps out a path in phase space.

And with these selected choices of m, b, and k, the path is a spiral. This reflects the fact that both x and x dot undergo a damped vibration. The system parameters can be set using these sliders at lower left. Let's suggest the mass to be 1/2 and leave k to be one, and watch what happens to the solutions as we vary the damping constant b. When b is small, the solution doesn't damp out as quickly, and when b is large, it damps out more quickly.

These things become much clearer when you think about them in terms of the roots of the characteristic polynomial of this equation. And we can display those roots by clicking the roots button here. What's shown here is the complex plane with the roots of the characteristic polynomial drawn on it. Also we have a readout of the two roots in green over here.

So let's watch what happens when we adjust b. I'm going to start with b equal to zero. In this case, there's no damping. The solutions are sinusoidal, the roots of the characteristic polynomial are purely imaginary, and the trajectory in phase space is an ellipse. Both the solution and its derivative vary sinusoidally.

As I increase b away from zero, I get spirals. And at the same time, the roots move off the imaginary axis. They acquire a negative real part, which is minus b over 2m, incidentally. And as I increase b further, the spiral opens up, the damping occurs more quickly, the roots move away from the imaginary axis. They increase the size of their real part. The damping occurs more quickly.

And you'll notice that they get closer and closer to the real axis as well. Their imaginary part is decreasing. That means that the circular frequency, the angular frequency of the solution, is decreasing. The period, or pseudo-period of the solution is decreasing.

You can see that in the graph, but it's much clearer to see what's happening to these roots of the characteristic polynomial. There are converging on the real axis, and eventually they merge into a double root on the real axis. This is critical damping.

From the readout, we see that the value of b seems to be about 1.41. Why don't you take a moment and calculate what b is for these values of m and k when critical damping occurs?

If we continue to increase b beyond critical damping, now the roots move out on the real axis. There are two exponential solutions, and the general solution is a linear combination of them. And there's no oscillation in any solution.